

Residuated Lattices

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Set notation

We assume a background of basic *set notation* and *logic*

A, B, C, \dots denote *sets*; $x \in A$ means x is an *element* of A

$\emptyset = \{\}$ is the *empty set*; *set-builder* notation: $\{x \mid P(x)\}$

Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ $\mathbb{N} = \text{Natural numbers}$

Union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ $\mathbb{Z} = \text{Integers}$
 $\mathbb{R} = \text{Real numbers}$

Set difference $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Ordered pairs $(x, y) = (u, v) \iff x = u \text{ and } y = v$

Cartesian product $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

Set of *n-tuples* $A^n = A \times A \times \dots \times A$ (n copies)

Outline

Part I

- Universal Algebra
- Examples of residuated lattices
- Congruences and normal filters
- The lattice of subvarieties
- Varieties generated by positive universal classes
- Direct decompositions and poset products

Part II

- Residuated Frames
- Decidability
- Enumerating finite residuated lattices

Algebras and subalgebras

An n -ary *operation* on a set A is a function $f : A^n \rightarrow A$

0-ary operations are *constants* (fixed elements of A)

An *algebra* $\mathbf{A} = (A, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \dots)$ is a set A with operations $f_i^{\mathbf{A}}$ of arity n_i

Superscript \mathbf{A} is useful when there are several algebras, otherwise omitted

The *type* of an algebra is the list of arities (n_1, n_2, \dots)

E.g. a *group* $\mathbf{G} = (G, \cdot, ^{-1}, 1)$ is an algebra of type $(2, 1, 0)$

Subsets: $B \subseteq A$ means for all x , if $x \in B$ then $x \in A$

$g = f|_B$ means for all $b_i \in B$, $g(b_1, \dots, b_n) = f(b_1, \dots, b_n) \in B$

\mathbf{B} is a *subalgebra* of \mathbf{A} if $B \subseteq A$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B$ (all i)

Homomorphisms and isomorphisms

Let \mathbf{A} , \mathbf{B} be algebras of the same type

A *homomorphism* $h : \mathbf{A} \rightarrow \mathbf{B}$ is a function $h : A \rightarrow B$ such that for all i

$$h(f_i^{\mathbf{A}}(a_1, \dots, a_{n_i})) = f_i^{\mathbf{B}}(h(a_1), \dots, h(a_{n_i}))$$

h is *onto* if $h[A] = \{h(a) \mid a \in A\} = B$

In this case $\mathbf{B} = h[\mathbf{A}]$ is called a *homomorphic image* of \mathbf{A}

h is *one-to-one* if for all $x, y \in A$, $x \neq y$ implies $h(x) \neq h(y)$

h is an *isomorphism* if h is a one-to-one and onto homomorphism

In this case \mathbf{A} is said to be *isomorphic* to \mathbf{B} , written $\mathbf{A} \cong \mathbf{B}$

Term algebras and equational classes

For a fixed type, the *terms with variables from a set* X is the smallest set $T(X)$ such that $X \subseteq T(X)$ and

if $t_1, \dots, t_{n_i} \in T(X)$ then " $f_i(t_1, \dots, t_{n_i})$ " $\in T(X)$ for all i

The *term-algebra over* X is $\mathbf{T}(X) = (T(X), f_1^{\mathbf{T}}, f_2^{\mathbf{T}}, \dots)$ with

$$f_i^{\mathbf{T}}(t_1, \dots, t_{n_i}) = "f_i(t_1, \dots, t_{n_i})" \quad \text{for all } i \text{ and } t_1, \dots, t_{n_i} \in T(X)$$

An *equation* is a pair of terms (s, t) written " $s=t$ "; often omit " $=$ "

An *assignment* into an algebra \mathbf{A} is a homomorphism $h : \mathbf{T}(X) \rightarrow \mathbf{A}$

An algebra \mathbf{A} *satisfies* $s=t$ if $h(s) = h(t)$ for all assignments into \mathbf{A}

For a set E of equations, $\text{Mod}(E) = \{\mathbf{A} \mid \mathbf{A} \text{ satisfies } s=t \text{ for all } s=t \in E\}$

An *equational class* is of the form $\text{Mod}(E)$ for some set of equations E

Products and HSP

The *union* of sets A_j ($j \in J$) is $\bigcup_{j \in J} A_j = \{x \mid x \in A_j \text{ for some } j \in J\}$

$f : J \rightarrow \bigcup_{j \in J} A_j$ is a *choice function* if $f(j) \in A_j$ for all $j \in J$

The *cartesian product* $\prod_{j \in J} A_j$ is the set of all choice functions

The *direct product* of algebras \mathbf{A}_j ($j \in J$) is $\mathbf{A} = \prod_{j \in J} \mathbf{A}_j$ where $A = \prod_{j \in J} A_j$ and $f_i^{\mathbf{A}}(a_1, \dots, a_{n_i})(j) = f_i^{\mathbf{A}_j}(a_1(j), \dots, a_{n_i}(j))$ for all $j \in J$

Let \mathcal{K} be a class of algebras of the same type

$H\mathcal{K}$ is the class of *homomorphic images* of members of \mathcal{K}

$S\mathcal{K}$ is the class of algebras isomorphic to *subalgebras* of members of \mathcal{K}

$P\mathcal{K}$ is the class of algebras isomorphic to *direct products* of members of \mathcal{K}

\mathcal{K} is a *variety* if $H(\mathcal{K}) = S(\mathcal{K}) = P(\mathcal{K}) = \mathcal{K}$ ($\stackrel{\text{Tarski}}{\iff} HSP(\mathcal{K}) = \mathcal{K}$)

Varieties and equational logic

Exercise: Show that every equational class is a variety

Theorem (Birkhoff 1935)

Every variety is an equational class

For a class \mathcal{K} of algebras $\text{Eq}(\mathcal{K}) = \{s=t \mid \mathbf{A} \text{ satisfies } s=t \text{ for all } \mathbf{A} \in \mathcal{K}\}$

An *equational theory* is of the form $\text{Eq}(\mathcal{K})$ for some class of algebras \mathcal{K}

$t[x \mapsto r]$ is the term t with all *occurrences* of x replaced by the term r

Theorem (Birkhoff 1935)

*E is an equational theory if and only if for all terms q, r, s, t
 $t=t \in E$; $s=t \in E \implies t=s \in E$; $r=s, s=t \in E \implies r=t \in E$
and $q=r, s=t \in E \implies s[x \mapsto q]=t[x \mapsto r] \in E$*

i.e. the rule of algebra: "replacing all x by equals in equals gives equals"

Examples of equational theories and varieties

A **binar** is an algebra (A, \cdot) with one binary operation $x \cdot y$, written xy

A **semigroup** is an **associative** binar, i.e. satisfies $(xy)z = x(yz)$

A **band** is an **idempotent** semigroup, i.e. satisfies $xx = x$

A **semilattice** is a **commutative** band, i.e. satisfies $xy = yx$

A **unital binar** is an algebra $(A, \cdot, 1)$ that satisfies $1x = x$ and $x1 = x$

A **monoid** is a unital binar that is associative, i.e. a unital semigroup

$(A, \cdot, {}^{-1}, 1)$ is a **group** if \cdot is associative, $1x = x$ and $x^{-1} \cdot x = 1$

Exercise: Show that a group satisfies $x1 = x$, $xx^{-1} = 1$ and $(x^{-1})^{-1} = x$

Hint: $x = 1x = (x^{-1})^{-1}x^{-1}x = (x^{-1})^{-1}1 = (x^{-1})^{-1}11 = (x^{-1})^{-1}x^{-1}x1 = 1x1 = x1$

Binary relations and partial orders

R is a **binary relation on a set** A if it is a subset of $A \times A$

E.g. the **identity relation** $id_A = \{(a, a) \mid a \in A\}$ is a binary relation on A

aRb means $(a, b) \in R$

R is **reflexive** if xRx for all $x \in A$

R is **antisymmetric** if xRy and yRx implies $x = y$

R is **transitive** if xRy and yRz implies xRz

R is a **partial order** if it is reflexive, antisymmetric and transitive

For a semilattice \mathbf{A} define $a \leq^{\mathbf{A}} b \iff ab = a$

Exercise: Prove that $\leq^{\mathbf{A}}$ is a partial order on A

Posets and meet-semilattices

A **poset** (A, \leq) is a set A with a partial order \leq on A

For $S \subseteq A$ the **meet** $\bigwedge S$ is defined by $x \leq \bigwedge S \iff x \leq s$ for all $s \in S$

Exercise: Prove $\bigwedge S$ is **unique** (if it exists; $\bigwedge =$ greatest lower bound)

$\bigwedge\{x, y\}$ is denoted by $x \wedge y$

Exercise: Prove that in a semilattice $ab = a \wedge b$ for the partial order $\leq^{\mathbf{A}}$

A **meet-semilattice** is a poset in which $a \wedge b$ exists for all a, b

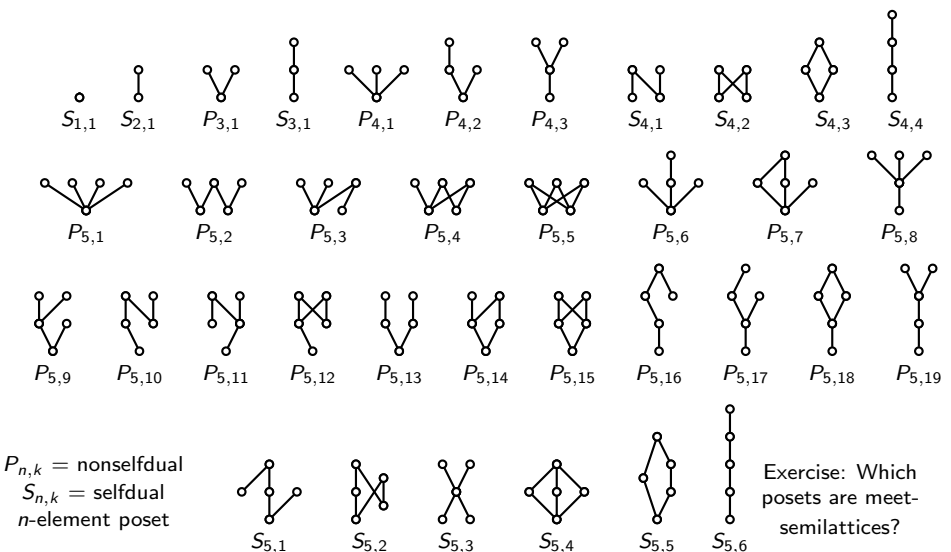
A meet-semilattice is **complete** if $\bigwedge S$ exists for all **nonempty** subsets S

Exercise: Prove if (A, \leq) is a meet-semilattice then (A, \wedge) is a semilattice

An element a has a **cover** b , denoted $a \prec b$, if $\{x \mid a \leq x \leq b\} = \{a, b\}$

A **Hasse diagram** of a poset has an upward line from dot a to b if $a \prec b$

(Dually)nonisomorphic connected posets with ≤ 5 elements



Lattices

For a relation \leq , define the *dual* \geq by $b \geq a \iff a \leq b$

$\mathbf{A}^\partial = (A, \geq)$ is the *dual poset* of $\mathbf{A} = (A, \leq)$

Every partial order concept has a dual, obtained by *interchanging* \leq and \geq

The *join* \vee is defined dually to the meet \wedge (join = least upper bound)

A *join-semilattice* is a poset where $a \vee b = \bigvee\{a, b\}$ exists for all a, b

A *lattice* is a poset that is a meet-semilattice and a join-semilattice

Note $x \leq y$ is *definable* by $x \vee y = y$, as well as by $x \wedge y = x$

Exercise: Show $\mathbf{A} = (A, \wedge, \vee)$ is a lattice iff \wedge, \vee are associative, commutative and *absorbtive*, i.e. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

Hint: $x \wedge x = x \wedge (x \vee (x \wedge y)) = x$

Examples of lattices

A lattice is *complete* if $\bigwedge S$ and $\bigvee S$ exist for all subsets S

A lattice is *bounded* if it has a top element \top and a bottom element \perp

Exercise: Show that every complete lattice is bounded

Exercise: Show any complete meet-semilattice with \top is a complete lattice

The *powerset* $\mathcal{P}(X)$ of all subsets of X is a complete lattice with \bigcap, \bigcup

The collection $\Lambda_{\mathcal{V}}$ of *subvarieties* of \mathcal{V} is a complete lattice with $\bigwedge = \bigcap$

Any *linear order* (i.e. $x \leq y$ or $y \leq x$ for all x, y) is a lattice

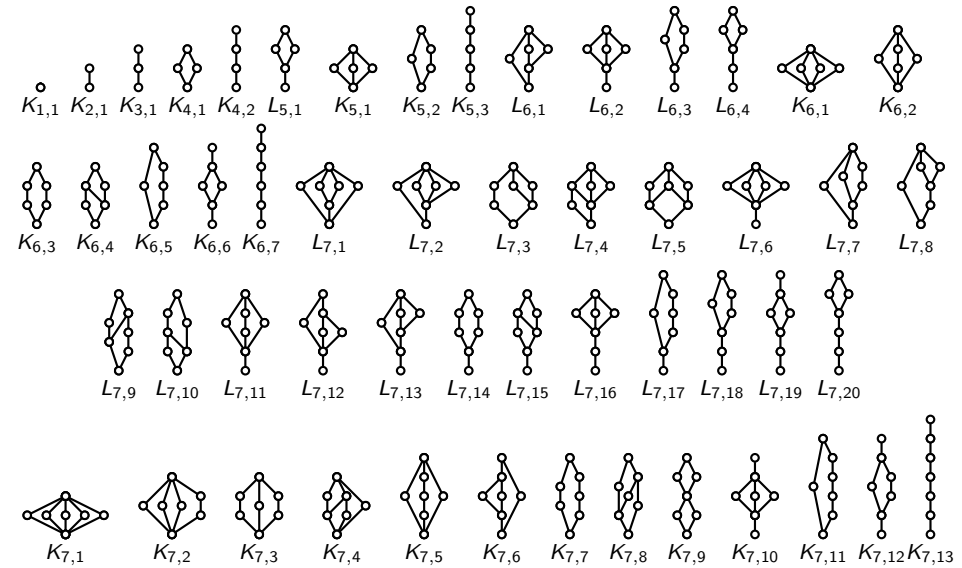
A lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds (\Leftrightarrow dual)

E.g. $\mathcal{P}(X)$ and any linear order are *distributive lattices*

All (dually-)nonisomorphic lattices with ≤ 7 elements

$K_{n,k}$ = selfdual lattice of size n

$L_{n,k}$ = nonselfdual lattices of size n



Equivalence relations and congruences

Let \mathbf{A} be an algebra and R a binary relation on A

R is *symmetric* if xRy implies yRx (implicitly quantified)

R is an *equivalence relation* if it is reflexive, symmetric and transitive

R is a *congruence* on \mathbf{A} if it is an equivalence relation and

$$xRy \text{ implies } f_i(a_1, \dots, x, \dots, a_n) R f_i(a_1, \dots, y, \dots, a_n) \quad (\text{all args, } i)$$

The set $\text{Con}(\mathbf{A})$ of *all congruences* on \mathbf{A} is a complete lattice with $\bigwedge = \bigcap$

$$\perp = id_A \text{ and } \top = A^2; \quad \text{con}(a, b) = \bigcap\{R \in \text{Con}(\mathbf{A}) \mid aRb\}$$

A *congruence class* is a set of the form $[a]_R = \{x \mid aRx\}$

$\{C_i : i \in I\}$ is a *partition* of A if $A = \bigcup_{i \in I} C_i$ and $C_i \cap C_j = \emptyset$ or $C_i = C_j$

The set A/R of *all congruence classes* is a partition of A

Homomorphic images and quotient algebras

The *quotient algebra* $\mathbf{A}/R = (A/R, f_1, f_2, \dots)$ is defined by

$$f_i([a_1]_R, \dots, [a_{n_i}]_R) = [f_i^{\mathbf{A}}(a_1, \dots, a_{n_i})]_R$$

Exercise: Show that f_i is well-defined if and only if R is a congruence

For a homom. $h : A \rightarrow B$, define the *kernel* $\ker h = \{(x, y) \mid h(x) = h(y)\}$

Exercise: Show that $\ker h$ is a congruence on \mathbf{A} and that

the *natural map* $[-]_R : \mathbf{A} \rightarrow \mathbf{A}/R$ is a homomorphism

Theorem (First Isomorphism Theorem)

$k : \mathbf{A}/\ker h \rightarrow h[\mathbf{A}]$ defined by $k([a]_{\ker h}) = h(a)$ is an isomorphism

Theorem (Second Isomorphism Theorem)

If $R \subseteq S$ are congruences on \mathbf{A} and $T = \{([a]_R, [b]_R) \mid aSb\}$ then $T \in \text{Con}(\mathbf{A}/R)$ and $(\mathbf{A}/R)/T \cong \mathbf{A}/S$

Birkhoff's Theorem says that every algebra is a subalgebra of a product of subdirectly irreducible algebras (s.i. algebras for short)

So the s.i. algebras are *building blocks* of varieties

For an element a in a poset, the *principal filter of* a is $\uparrow a = \{x \mid a \leq x\}$

A subset S of a poset is an *upset* if for all $a \in S$ we have $\uparrow a \subseteq S$

S is *down-directed* if for all $a, b \in S$ there is a $c \in S$ with $c \leq a$ and $c \leq b$

A *filter* F is a down-directed upset

An *ideal* is the dual concept of a filter, i.e. an *up-directed downset*

Exercise: Show that the 2-element semilattice is the only s.i. semilattice and that the 2-element lattice is the only s.i. *distributive* lattice

Hint: Every congruence is the intersection of congruences with two blocks

Every semilattice is a subalgebra of a product of two-element semilattices

Subdirectly irreducible algebras

An algebra is *directly decomposable* if it is isomorphic to a direct product of nontrivial algebras (happens rarely)

Let $R_j \in \text{Con}(\mathbf{A})$ and define $h : \mathbf{A} \rightarrow \prod_{j \in J} \mathbf{A}/R_j$ by $h(a)(j) = [a]_{R_j}$

Exercise: Show that h is one-to-one if and only if $\bigcap_{j \in J} R_j = id_{\mathbf{A}}$

In this case h is called a *subdirect decomposition* of \mathbf{A}

An element c in a complete lattice is *completely meet irreducible* if $c = \bigwedge S$ implies $c \in S$ for all subsets S ; equiv. if c has a *unique* cover

\mathbf{A} is *subdirectly irreducible* if $id_{\mathbf{A}}$ is completely meet irreducible in $\text{Con}(\mathbf{A})$

Theorem (Birkhoff 1944)

Every algebra \mathbf{A} has a subdirect decomposition using only subdirectly irreducible homomorphic images of \mathbf{A}

Introductory references for further background reading

Garrett Birkhoff, "**Lattice Theory**", 3rd ed., AMS Colloquium Publications, Vol. 25, 1967

Stan Burris and H. P. Sankappanavar, "**A Course in Universal Algebra**", Springer-Verlag, 1981, online at www.math.uwaterloo.ca/~snburris/

Brian Davey and Hilary Priestley, "**Introduction to Lattices and Order**", 2nd ed, Cambridge University Press, 2002

Nick Galatos, Peter Jipsen, Tomasz Kowalski and Hiroakira Ono, "**Residuated Lattices: an algebraic glimpse at substructural logics**", Studies in Logics and the Foundations of Mathematics, Elsevier, 2007

Residuated maps

A function $f : (A, \leq) \rightarrow (B, \leq)$ is *residuated* if some $g : B \rightarrow A$ satisfies

$$f(x) \leq y \iff x \leq g(y) \quad \text{for all } x \in A, y \in B$$

g is called the *residual of f* and $g(y) = \max\{x \mid f(x) \leq y\}$ (if it exists)

Exercise: f preserves all existing joins and g preserves all existing meets

Exercise: Show that f is residuated $\iff f, g$ are order-preserving and

$$f(g(y)) \leq y \quad \text{and} \quad x \leq g(f(x)) \quad \text{for all } x \in A, y \in B$$

Exercise: If \mathbf{A}, \mathbf{B} are lattices then f is residuated with residual $g \iff$

$$f(x \wedge g(y)) \vee y = y \quad \text{and} \quad x = x \wedge g(f(x) \vee y) \quad \text{hold}$$

Hence residuation can be expressed by *equations* (also in semilattices)

Algebras of relations

For binary relations R and S on a set X , we denote by

- R' the complement $X^2 - R$ and by R^\smile the *converse* $\{(y, x) \mid xRy\}$
- $R \cdot S$ the *composition* $\{(x, y) \mid (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z\}$
- $R \setminus S = (R^\smile \cdot S')$ and $S / R = (S' \cdot R^\smile)'$
- $R \rightarrow S = R' \cup S$

Exercise: Check that

- $(\mathcal{P}(X^2), \cap, \cup, \rightarrow, \emptyset, X^2)$ is a Boolean algebra
- $(\mathcal{P}(X^2), \cdot, id_X)$ is a monoid
- for all $R, S, T \subseteq X^2$,

$$R \cdot S \subseteq T \iff S \subseteq R \setminus T \iff R \subseteq T / S$$

Heyting algebras and Boolean algebras

$\mathbf{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a *Heyting algebra* (HA) if

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and
- for all $a, b, c \in A$, $a \wedge b \leq c \iff b \leq a \rightarrow c$ (\wedge -residuation)

The unary *negation* operation \neg is defined by $\neg x = x \rightarrow 0$

A *Boolean algebra* (BA) is a Heyting algebra that satisfies $\neg \neg x = x$

Exercise: Heyting algebras are distributive lattices and $x \wedge \neg x = 0$

BAs are also defined by $x \vee \neg x = 1$, i.e. \neg is a *complement*

\wedge -residuation can be written equationally (hint: $a \wedge _$ has residual $a \rightarrow _$)

BAs give algebraic semantics for *classical propositional logic* $x \rightarrow y = \neg x \vee y$

HAs give algebraic semantics for *intuitionistic propositional logic*

Relation algebras

A *relation algebra* is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, \setminus, /, 1, ')$ with

- $x \rightarrow y = x' \vee y$, $\perp = 1 \wedge 1'$ and $\top = 1 \vee 1'$ such that
- $(A, \wedge, \vee, \rightarrow, \perp, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid
- for all $a, b, c \in A$,

$$a \cdot b \leq c \iff b \leq a \setminus c \iff a \leq c / b \quad (\text{residuation})$$

- $x^{\smile\smile} = x$ and $(x \cdot y)^\smile = y^\smile \cdot x^\smile$ where x^\smile is defined as $(x \setminus 1)'$

Exercise: Show that $x^\smile = (1' / x)'$ and $x^\smile(xy)' \leq y'$

ℓ -groups

A lattice-ordered group is a lattice with an order-pres. group operation

Alternatively, a *lattice-ordered group* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- (L, \wedge, \vee) is a lattice
- $(L, \cdot, 1)$ is a monoid
- for all $a, b, c \in L$,

$$a \cdot b \leq c \iff b \leq a \backslash c \iff a \leq c / b$$

- and \mathbf{L} satisfies $(1/x) \cdot x = 1$

Exercise: Show that $(L, \cdot, {}^{-1}, 1)$ is a group, where $x^{-1} = 1/x = x \backslash 1$

Example. The reals \mathbb{R} under the usual order, addition and subtraction

Powerset of a monoid

Let $\mathbf{M} = (M, \cdot, e)$ be a monoid and $X, Y \subseteq M$ and define

$$X \cdot Y = \{x \cdot y \mid x \in X \text{ and } y \in Y\}$$

$$X \backslash Y = \{z \in M \mid X \cdot \{z\} \subseteq Y\}$$

$$Y / X = \{z \in M \mid \{z\} \cdot X \subseteq Y\}$$

Exercise: Show that the powerset $\mathcal{P}(M)$ satisfies

- $(\mathcal{P}(M), \cap, \cup)$ is a lattice
- $(\mathcal{P}(M), \cdot, \{e\})$ is a monoid
- for all $X, Y, Z \subseteq M$,

$$X \cdot Y \subseteq Z \iff Y \subseteq X \backslash Z \iff X \subseteq Z / Y$$

Ideals of a ring

Let \mathbf{R} be a ring with unit and let $\mathcal{I}(\mathbf{R})$ be the set of (two-sided) ideals of \mathbf{R}

For $I, J \in \mathcal{I}(\mathbf{R})$ define $I \cdot J = \{a_1 b_1 + \dots + a_n b_n \mid a_i \in I, b_i \in J\}$

$$I \backslash J = \{a \in R \mid Ia \subseteq J\} \quad J / I = \{a \in R \mid aI \subseteq J\}$$

$$I \vee J = \{a + b \mid a \in I \text{ and } b \in J\}$$

Exercise: Show that the set of ideal $\mathcal{I}(\mathbf{R})$ is closed under $\backslash, /$ and

- $(\mathcal{I}(\mathbf{R}), \cap, \vee)$ is a lattice
- $(\mathcal{I}(\mathbf{R}), \cdot, R)$ is a monoid
- for all ideals I, J, K of \mathbf{R}

$$I \cdot J \subseteq K \iff J \subseteq I \backslash K \iff I \subseteq K / J$$

Residuated lattices

A *residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- (L, \wedge, \vee) is a lattice
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$, $ab \leq c \iff b \leq a \backslash c \iff a \leq c / b$

A **Full Lambek algebra** is a residuated lattice with a new constant 0

In an FL-algebra, define two *linear negations* $\sim x = x \backslash 0$ and $-x = 0 / x$

A FL-algebra or residuated lattice is called

- **commutative** if $(L, \cdot, 1)$ is commutative ($xy = yx$)
- **distributive** if (L, \wedge, \vee) is distributive
- **integral** if it satisfies $x \leq 1$
- **contractive** if it satisfies $x \leq x^2$
- **involutive** if it satisfies $\sim -x = x = -\sim x$

Properties

- 1 $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$
- 2 $x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z)$ and $(y \wedge z) / x = (y / x) \wedge (z / x)$
- 3 $x / (y \vee z) = (x / y) \wedge (x / z)$ and $(y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x)$
- 4 $(x / y)y \leq x$ and $y(y \setminus x) \leq x$
- 5 $x(y / z) \leq (xy) / z$ and $(z \setminus y)x \leq z \setminus (yx)$
- 6 $(x / y) / z = x / (zy)$ and $z \setminus (y \setminus x) = (yz) \setminus x$
- 7 $x \setminus (y / z) = (x \setminus y) / z$
- 8 $x / 1 = x = 1 \setminus x$
- 9 $1 \leq x / x$ and $1 \leq x \setminus x$
- 10 $x \leq y / (x \setminus y)$ and $x \leq (y / x) \setminus y$
- 11 $y / ((y / x) \setminus y) = y / x$ and $(y / (x \setminus y)) \setminus y = x \setminus y$
- 12 $x / (x \setminus x) = x$ and $(x / x) \setminus x = x$
- 13 $(z / y)(y / x) \leq z / x$ and $(x \setminus y)(y \setminus z) \leq x \setminus z$

Multiplication is order preserving in both arguments. Each division operation is order preserving in the **numerator** and order reversing in the **denominator**.

Lattice/monoid properties

$$(z / y)(y / x)x \leq (z / y)y \leq z \Rightarrow (z / y)(y / x) \leq z / x$$

RL's satisfy **no special purely** lattice-theoretic or monoid-theoretic property

Every lattice can be embedded in a (cancellative) residuated lattice

Every monoid can be embedded in a (distributive) residuated lattice

Proofs of some of these properties

$$\begin{aligned} x(y \vee z) \leq w &\Leftrightarrow y \vee z \leq x \setminus w \\ &\Leftrightarrow y \leq x \setminus w \text{ and } z \leq x \setminus w \\ &\Leftrightarrow xy \leq w \text{ and } xz \leq w \\ &\Leftrightarrow xy \vee xz \leq w \end{aligned}$$

$$x / y \leq x / y \Rightarrow (x / y)y \leq x$$

$$x(y / z)z \leq xy \Rightarrow x(y / z) \leq (xy) / z$$

$$[(x / y) / z](zy) \leq x \Rightarrow (x / y) / z \leq x / (zy)$$

$$[x / (zy)]zy \leq x \Rightarrow x / (zy) \leq (x / y) / z$$

$$\begin{aligned} w \leq x \setminus (y / z) &\Leftrightarrow xw \leq y / z \\ &\Leftrightarrow xwz \leq y \\ &\Leftrightarrow wz \leq x \setminus y \\ &\Leftrightarrow w \leq (x \setminus y) / z \end{aligned}$$

Congruences in groups and Boolean algebras

Recall that a **congruence** on an algebra **A** is an equivalence relation on **A** that is compatible with the operations of **A**

A subgroup **N** of a group **G** is **normal** if $a \in N$ implies $x^{-1}ax \in N$

Congruences in **groups** correspond to **normal subgroups**:

For a congruence **R** on **G**, the congruence class $[1]_R$ is a normal subgroup

Given a normal subgroup **N** of a group **G**, the relation R_N is a congruence, where $(a, b) \in R_N$ iff $a \setminus b \in N$ iff $\{a \setminus b, b \setminus a\} \subseteq N$

Congruences in **Boolean algebras** correspond to **filters**:

Given a congruence **R** on a BA, the congruence class $[1]_R$ is a filter of **A**

Given a filter **F** of a Boolean algebra **A**, R_F is a congruence, where $(a, b) \in R_F$ iff $a \leftrightarrow b \in F$ iff $\{a \rightarrow b, b \rightarrow a\} \subseteq F$

Congruences on **rings** correspond to (two-sided) **ideals**

Congruences on **ℓ -groups** correspond to **convex ℓ -subgroups**

Congruences on **monoids** do not correspond to any particular kind of subset

Do congruences on **residuated lattices** correspond to certain subsets?

Correspondence

If R is a congruence on \mathbf{A} and F is a normal filter of \mathbf{A} then

- $\mathbf{F}(R) = \uparrow[1]_R$ is a normal filter of \mathbf{A} and
- $\mathbf{R}(F) = \{(a, b) \mid a \setminus b, b \setminus a \in F\}$ is a congruence of \mathbf{A}

Normality of $\mathbf{F}(R)$: If $a \geq c \in [1]_R$ and $b \in A$ then

$$ba/b \geq (bc/b \wedge 1) \quad R \quad (b1/b \wedge 1) = 1$$

Compatibility of $\mathbf{R}(F)$: If $a \setminus b, b \setminus a \in F$ then

$$(a \wedge c)(a \setminus b \wedge 1) \leq a(a \setminus b) \wedge c1 \leq b \wedge c \text{ so } (a \wedge c) \setminus (b \wedge c) \in F \text{ (}\forall \text{ same)}$$

$$a \setminus b \leq ca \setminus cb \in F, \quad c \setminus (a \setminus b)c \leq ac \setminus bc \in F \text{ and } a \setminus b \leq (c \setminus a) \setminus (c \setminus b) \in F$$

$$a \setminus b \leq (a \setminus c) / (b \setminus c) \in F \text{ so } (b \setminus c) \setminus ((a \setminus c) / (b \setminus c))(b \setminus c) \leq (a \setminus c) \setminus (b \setminus c) \in F$$

Let \mathbf{A} be a residuated lattice and $a, x \in A$. We define the **conjugates** $\lambda_a(x) = [a \setminus (xa)] \wedge 1$ and $\rho_a(x) = ax/a \wedge 1$

An **iterated conjugate** is a composition $\gamma_{a_1}(\gamma_{a_2}(\dots \gamma_{a_n}(x)))$, where $n \in \omega$, $a_1, a_2, \dots, a_n \in A$ and $\gamma_{a_i} \in \{\lambda_{a_i}, \rho_{a_i}\}$, for all i

F is a **filter** of \mathbf{A} if it is a lattice filter, $a, b \in F$ implies $ab \in F$, and $1 \in F$

A filter is called **normal** if $x \in F$ implies $ax/a, a \setminus xa \in F$ for all $a \in A$ (i.e. it is closed under all conjugates; also called a **deductive filter**)

We will consider the correspondence between:

- **Congruences** on \mathbf{A} and
- **Normal filters** of \mathbf{A}

There are also correspondences with e.g. convex normal subalgebras

The normal filter lattice is isomorphic to $\mathbf{Con}(\mathbf{A})$

The normal filters of \mathbf{A} form a lattice, denoted by $\mathbf{NFil}(\mathbf{A})$

This lattice is isomorphic to the lattice $\mathbf{Con}(\mathbf{A})$ via the maps \mathbf{F} and \mathbf{R}

Claim: \mathbf{F} and \mathbf{R} are inverse maps

$\mathbf{F} = \uparrow[1]_{\mathbf{R}(F)}$: $a \in F$ implies $a \wedge 1 \in F$. Also $a \wedge 1 \leq 1$ implies $1 \leq (a \wedge 1) \setminus 1 \in F$, hence $a \wedge 1 \in [1]_{\mathbf{R}(F)}$.

$a \in \uparrow[1]_{\mathbf{R}(F)}$ implies $a \geq b \in [1]_{\mathbf{R}(F)}$, hence $1 \setminus b = b \in F$ so $a \in F$.

$\mathbf{R} = \mathbf{R}(\uparrow[1]_R)$: If $aR(\uparrow[1]_R)b$, then $(a \setminus b) \wedge 1 \in [1]_R$ since $[1]_R$ is convex. Hence $a \vee b R a(a \setminus b \wedge 1) \vee b = b$, and similarly $a \vee b R a$, so aRb .

If aRb , then $1 = (a \setminus a \wedge b \setminus b \wedge 1)R(a \setminus b \wedge b \setminus a \wedge 1)$, hence $aR(\uparrow[1]_R)b$

- The normal filter $F(X)$ of \mathbf{A} generated by $X \subseteq A$ is $\uparrow\{\gamma_1(x_1)\gamma_2(x_2)\cdots\gamma_n(x_n) \mid n \geq 0, x_i \in X, \gamma_i \text{ iterated conjugates}\}$
- The congruence $R(Y)$ on \mathbf{A} generated by $Y \subseteq A^2$ is $\mathbf{R}(F(\{a \setminus b \wedge b \setminus a \wedge 1 \mid (a, b) \in Y\}))$

For a **finite** residuated lattice any filter is generated by an idempotent ≤ 1

An element c is **central** if $cx = xc$ for all $x \in A$

The **normal filters** are the ones generated by a central idempotent ≤ 1

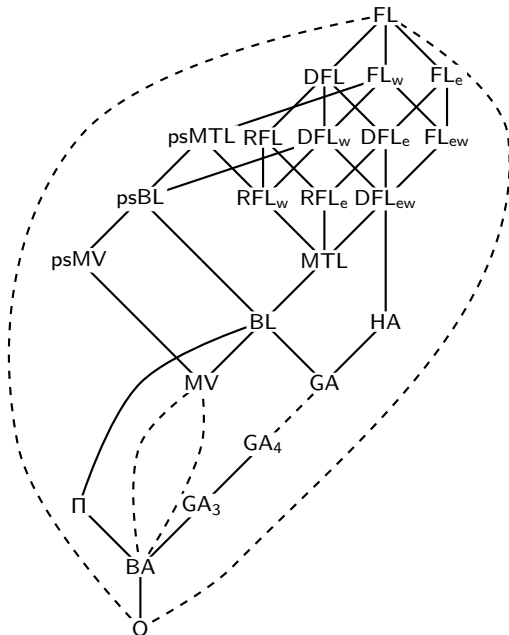
\mathbf{A} is **subdirectly irreducible** if it has a maximal central idempotent < 1

Exercise: A finite RL $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ with \mathbf{B}, \mathbf{C} nontrivial if and only if there are central idempotents $b, c < 1$ such that $b \wedge c = \perp$ and $b \vee c = 1$

RL and FL denote the variety of **residuated lattices** and **FL-algebras** resp.

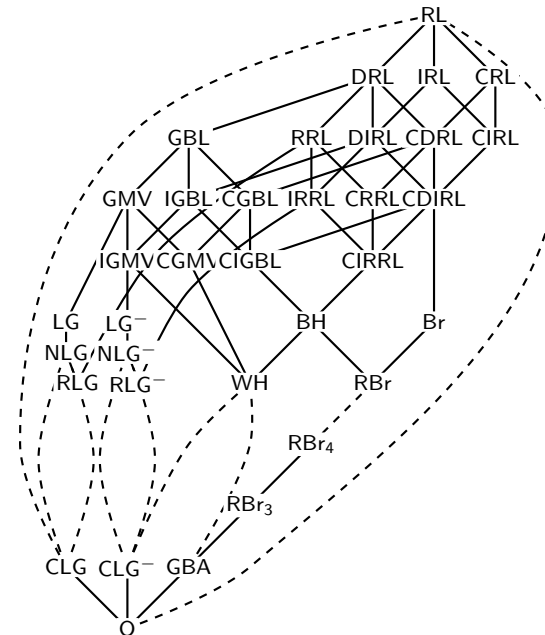
We view RL as the subvariety of FL axiomatized by $0 = 1$

Let's take a look at a map of the subvariety lattices Λ_{FL} and Λ_{RL}



Some subvarieties of FL ordered by inclusion

- D = distributive
- w = bounded by 0, 1
- e = commutative
- R = representable
- ps = pseudo
- MTL = monoidal t-norm
- BL = basic logic algebras
- $x \wedge y = x(x \rightarrow y)$
- HA = Heyting algebras
- GA = Gödel algebras
- MV = Multivalued algs
- $x \vee y = (x \rightarrow y) \rightarrow y$
- Π = Product algebras
- BA = Boolean algebras
- O = Trivial algebras



Some subvarieties of RL ordered by inclusion

- D = distributive
- I = integral $x \leq 1$
- C = commutative
- R = representable
- GBL = generalized BL
- $x \wedge y = x(x \setminus (x \wedge y))$ & mi
- GMV = generalized MV
- $x \vee y = x / ((x \vee y) \setminus x)$ & mi
- BH = basic hoops
- Br = Brouwerian algebras
- WH = Wajsberg hoops
- LG = lattice-ordered groups
- N = normal valued
- O = Trivial algebras

Ultraproducts

\mathcal{F} is a *filter over a set J* if \mathcal{F} is a filter in $(\mathcal{P}(J), \subseteq)$

\mathcal{F} defines a *congruence* on $\prod_{j \in J} \mathbf{A}_j$ via $a \equiv_{\mathcal{F}} b \Leftrightarrow \{j \in J \mid a(j) = b(j)\} \in \mathcal{F}$

$\prod_{j \in J} \mathbf{A}_j / \equiv_{\mathcal{F}}$ is called an *ultraproduct* if J is a maximal filter

$P_u \mathcal{K}$ is the *class of all ultraproducts* of members of \mathcal{K}

\mathcal{K} is *finitely axiomatizable* if $\mathcal{K} = \text{Mod}(E)$ for a finite set E

Theorem

\mathcal{K} and $P_u \mathcal{K}$ satisfy the same first-order formulas

If \mathcal{K} is a finite class of finite algebras then $P_u \mathcal{K} = \mathcal{K}$

Define $V(\mathcal{K}) = \text{HSP}(\mathcal{K})$, and recall this is a variety (i.e. closed via H, S, P)

Exercise: Show that $V(\mathcal{K})$ is the smallest variety containing \mathcal{K}

Congruence distributivity and Jónsson's Theorem

\mathbf{A} is *congruence distributive* (CD) if $\text{Con}(\mathbf{A})$ is a distributive lattice

A class \mathcal{K} of algebras is *CD* if every algebra in \mathcal{K} is CD

Theorem (Jónsson 1967)

If $\mathcal{V} = V(\mathcal{K})$ is congruence distributive then $\mathcal{V}_{S1} \subseteq \text{HSP}_u \mathcal{K}$

Theorem

If \mathcal{K} is a finite class of finite algebras and $V(\mathcal{K})$ is CD then $\mathcal{V}_{S1} \subseteq \text{HSK}$

If $\mathbf{A}, \mathbf{B} \in \mathcal{V}_{S1}$ are finite nonisomorphic and \mathcal{V} is CD then $V(\mathbf{A}) \neq V(\mathbf{B})$

\mathcal{V} is *finitely generated* if $\mathcal{V} = V(\mathcal{K})$ for some finite class of finite algebras

Theorem

A finitely generated CD variety has only finitely many subvarieties

Lattices of subvarieties

$\mathbf{A}' = (A, f_{i_1}, f_{i_2}, \dots)$ is a *reduct* of $\mathbf{A} = (A, f_1, f_2, \dots)$ if $i_1 < i_2 < \dots$ in \mathbb{N}

In this case \mathbf{A} is called an *expansion* of \mathbf{A}'

Exercise: If \mathbf{A}' is a reduct of \mathbf{A} then $\text{Con}(\mathbf{A})$ is a sublattice of $\text{Con}(\mathbf{A}')$

The variety of lattices is CD, so any variety of lattice expansions is CD

Recall that for a variety \mathcal{V} the lattice of subvarieties is denoted by $\Lambda_{\mathcal{V}}$

Theorem

$\text{HSP}_u(\mathcal{K} \cup \mathcal{L}) = \text{HSP}_u \mathcal{K} \cup \text{HSP}_u \mathcal{L}$ for any classes \mathcal{K}, \mathcal{L}

If \mathcal{V} is CD then $\Lambda_{\mathcal{V}}$ is distributive and the map $\mathcal{V} \mapsto \mathcal{V}_{S1}$ is a lattice embedding of $\Lambda_{\mathcal{V}}$ into $\mathcal{P}(\mathcal{V}_{S1})$ (unless \mathcal{V}_{S1} is a proper class)

Size of the lattice of subvarieties

A set S is *countable* if there is a one-to-one function $f : S \rightarrow \mathbb{N}$

The subvariety lattices of **HA** (Heyting algebras) and **Br** (Brouwerian algebras) are *uncountable*, hence so are Λ_{FL} and Λ_{RL} .

We will

- determine the *size* of the set of atoms in Λ_{FL} .
- outline a method for finding *axiomatizations* of certain varieties
- give a description of *joins* in Λ_{FL} .

The variety BA of Boolean algebras is generated by the 2-element alg. **2**

I.e. $BA = HSP(\mathbf{2}) = V(\mathbf{2})$

Proof idea: In a distributive lattice, any maximal proper congruence has two classes: a (prime) filter and a (prime) ideal, and each pair of distinct elements can be separated by such a congruence (the prime ideal-filter theorem for distributive lattices)

Then show that every BA has a subdirect decomposition with copies of **2**

So $BA = SP(\mathbf{2}) = HSP(\mathbf{2})$

Now, $HSP_U(\mathbf{2}) = \{\mathbf{2}, \mathbf{1}\}$, hence $(V(\mathbf{2}))_{SI} = \{\mathbf{2}\}$

Cancellative atoms

Left cancellativity ($ab = ac \Rightarrow b = c$) is equational: $x \setminus (xy) = y$

Right cancellativity is $(yx)/x = y$

CanRL denotes the variety of (left and right) cancellative RLs

Theorem

There are only 2 cancellative atoms: $V(\mathbb{Z})$ and $V(\mathbb{Z}^-)$

Let $\mathbf{L} \in \text{CanRL}$. For $a \leq 1$, we have $1 \leq 1/a$.

Claim: If $a < 1$ and $1/a = 1$, then the subalgebra generated by a is \mathbb{Z}^-

Since $a < 1$, we get $a^{n+1} < a^n$, for all $n \in \mathbb{N}$, by order preservation and cancellativity

Moreover, $a^{k+m}/a^m = a^k$ and $a^m/a^{m+k} = 1$, for all $m, k \in \mathbb{N}$

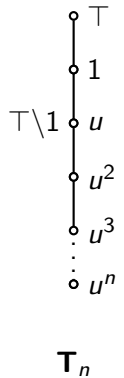
Consider the residuated chain \mathbf{T}_n defined to the right

We also let $\top u = u\top = u$, hence $\top \setminus 1 = u$

Note that \mathbf{T}_n is *strictly simple*, i.e. has no non-trivial subalgebras or homomorphic images

It follows that $V(\mathbf{T}_n)$ is an atom of \mathbf{A}_{RL}

Moreover, all these atoms are distinct so \mathbf{A}_{RL} has **at least countably many** atoms



$\mathbb{Z}^- = \{n \in \mathbb{Z} \mid n \leq 0\}$ is a cancellative RL with min, max, + as operations

Claim: If for all $x < 1$, we have $1 < 1/x$, then \mathbf{L} is an ℓ -group.

For $a \in L$ set $x = (1/a)a$. Note that $x \leq 1$, and if $x < 1$, then $1/x = 1/(1/a)a = (1/a)/(1/a) = 1$, cancellativity; so $x = 1$.

The construction of \mathbb{Z}^- from \mathbb{Z} actually works in general

The *negative cone* of a RL $\mathbf{A} = (A, \wedge, \vee, \cdot, \setminus, /, 1)$ is the RL $\mathbf{A}^- = (A^-, \wedge, \vee, \cdot, \setminus^{A^-}, /^{A^-}, 1)$, where $A^- = \{a \in A : a \leq 1\}$, $a \setminus^{A^-} b = (a \setminus b) \wedge 1$ and $b /^{A^-} a = (b/a) \wedge 1$.

Idempotent atoms generated by chains

For $S \subseteq \mathbb{Z}$, we define

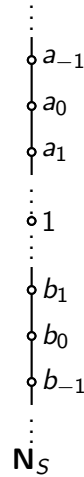
$a_i b_i = a_i$, if $i \in S$ and

$a_i b_i = b_i$, if $i \notin S$.

Although we may have

- $S \neq T$, but $\mathbf{N}_S \cong \mathbf{N}_T$
- $\mathbf{N}_S \not\cong \mathbf{N}_T$, but $V(\mathbf{N}_S) = V(\mathbf{N}_T)$
- $V(\mathbf{N}_S)$ is not an atom

N. Galatos [2004] proved that there are **continuum many** atoms $V(\mathbf{N}_S)$



Representable RL's

A residuated lattice is called **representable** if it is a subdirect product of linearly ordered RL's

RRL denotes the class of representable RL's

A linearly ordered RL satisfies the first-order formula $(\forall x, y)(x \leq y \text{ or } y \leq x)$, i.e., $[(\forall x, y)(1 \leq x \setminus y \text{ or } 1 \leq y \setminus x)]$

Representable Heyting algebras form a variety axiomatized by $1 = (x \rightarrow y) \vee (y \rightarrow x)$ (= Gödel algebras)

Representable commutative RL's form a variety axiomatized by $1 \leq (x \rightarrow y) \vee (y \rightarrow x)$

RRL is a variety axiomatized by $1 = \gamma_1(x \setminus y) \vee \gamma_2(y \setminus x)$

Goal: Given a class \mathcal{K} of RL's axiomatized by a set of positive universal first-order formulas (**PUF's**), provide an axiomatization for $V(\mathcal{K})$

Joins of varieties

Recall that the meet of two varieties in \mathbf{A}_{FL} is their intersection

In fact, if $\mathcal{V}_1 = \text{Mod}(E_1)$, $\mathcal{V}_2 = \text{Mod}(E_2)$, then $\mathcal{V}_1 \wedge \mathcal{V}_2 = \text{Mod}(E_1 \cup E_2)$

But the join of two varieties is the variety **generated** by their union

Also, if \mathcal{V}_1 is axiomatized by E_1 and \mathcal{V}_2 by E_2 , then $\mathcal{V}_1 \vee \mathcal{V}_2$ is usually **not** axiomatized by $E_1 \cap E_2$.

Goals

- Find an **axiomatization** of $\mathcal{V}_1 \vee \mathcal{V}_2$ in terms of E_1 and E_2
- Find situations where: if E_1 and E_2 are finite, then $\mathcal{V}_1 \vee \mathcal{V}_2$ is **finitely axiomatized**

Finite basis

[K. Baker 1977, B. Jónsson 1979] If \mathcal{V} is a congruence distributive variety of finite type and \mathcal{V}_{FSI} is strictly elementary, then \mathcal{V} is **finitely axiomatized**.

Strictly elementary: Axiomatized by a single first-order sentence

Finitely SI: $id_{\mathbf{A}}$ is not the intersection of two non-trivial congruences

Cor. For every variety \mathcal{V} of RL's, if \mathcal{V}_{FSI} is **strictly** elementary, then the finitely axiomatized subvarieties of \mathcal{V} form a lattice

Pf. For finitely axiomatized subvarieties $\mathcal{V}_1, \mathcal{V}_2$, $(\mathcal{V}_1 \vee \mathcal{V}_2)_{FSI} = (\mathcal{V}_1 \cup \mathcal{V}_2)_{FSI}$ is strictly elementary.

Let $\mathcal{V}_1, \mathcal{V}_2$ be subvarieties of RL axiomatized by E_1, E_2 , respectively, where E_1, E_2 have *no variables in common*.

The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the universal closure of $(\text{AND } E_1)$ or $(\text{AND } E_2)$, over infinitary logic, which is equivalent to the set $\{\forall \forall (\varepsilon_1 \text{ or } \varepsilon_2) : \varepsilon_1 \in E_1, \varepsilon_2 \in E_2\}$ of **positive universal first-order formulas (PUFs)**.

In a RL, we say that 1 is *weakly join irreducible*, if for all negative a, b , whenever $1 = \gamma(a) \vee \gamma'(b)$, for all iterated conjugates γ, γ' , then $a = 1$ or $b = 1$

Theorem (Galatos 2004)

A residuated lattice is FSI iff 1 is weakly join-irreducible

Every PUF is equivalent to (the universal closure of) a disjunction of conjunctions of equations

$$s = t \iff (s \leq t \text{ and } t \leq s) \iff (1 \leq s \setminus t \text{ and } 1 \leq t \setminus s)$$

PUF and equations

Thm. For a PUF α and a FSI RL \mathbf{A} , $\mathbf{A} \models \alpha$ iff $\mathbf{A} \models \tilde{\alpha}$. **Pf.** (\Rightarrow) If \bar{a} are elements in A , then $1 \leq r_i(\bar{a})$ for some i .

So, $\gamma(r_i(\bar{a})_{\wedge 1}) = 1$, for all γ ; hence, $\gamma_1(r_1(\bar{a})_{\wedge 1}) \vee \cdots \vee \gamma_k(r_k(\bar{a})_{\wedge 1}) = 1$.

(\Leftarrow) We have $1 = \gamma_1(r_1(\bar{a})_{\wedge 1}) \vee \cdots \vee \gamma_k(r_k(\bar{a})_{\wedge 1})$, for all γ_i .

Since \mathbf{A} is FSI, 1 is weakly join irreducible, so $r_i(\bar{a})_{\wedge 1} = 1$, for some i ; i.e., $r_i(\bar{a}) \leq 1$.

$$\alpha = \forall \bar{x} (1 \leq r_1 \text{ or } \cdots \text{ or } 1 \leq r_k)$$

$$\tilde{\alpha} = \{\gamma_1 \vee \cdots \vee \gamma_k = 1 \mid \gamma_i \in \Gamma_Y(r_i)\}$$

Theorem

Let \mathcal{K} be a class of RLs axiomatized by a set Ψ of PUF. Then $\mathcal{V}(\mathcal{K})$ is axiomatized, relative to RL, by $\tilde{\Psi}$.

Every conjunction of equations $1 \leq p_i$ is equivalent to the equation $1 \leq p_1 \wedge \cdots \wedge p_n$

So, every PUF is equivalent to a formula of the form

$$\alpha = \forall \bar{x} (1 \leq r_1 \text{ or } \cdots \text{ or } 1 \leq r_k)$$

$$\text{Let } \tilde{\alpha}_0 \text{ be } (r_1 \wedge 1) \vee \cdots \vee (r_k \wedge 1) = 1$$

Also, for $m > 0$ and \aleph_0 fresh variables Y , we define $\tilde{\alpha}_m$ as the set of all equations of the form

$$\gamma_1 \vee \cdots \vee \gamma_k = 1$$

where $\gamma_i \in \Gamma_Y^m(r_i)$ for each $i \in \{1, \dots, k\}$. Set $\tilde{\alpha} = \bigcup_{n \in \omega} \tilde{\alpha}_n$

Here $\Gamma_Y^m(a) = \{\pi_{y_1} \pi_{y_2} \cdots \pi_{y_m}(a \wedge 1) \mid y_i \in Y, \pi_{y_i} \in \{\lambda_{y_i}, \rho_{y_i}\}\}$

Proof.

Let $\mathbf{A} \in \text{RL}_{\text{SI}}$. By congruence distributivity and Jónsson's Lemma, $\mathbf{A} \in \mathcal{V}(\mathcal{K})$ iff $\mathbf{A} \in \text{HSP}_{\text{U}}(\mathcal{K})$. Since PUFs are preserved under H, S and P, $\mathbf{A} \in \text{HSP}_{\text{U}}(\mathcal{K})$ iff $\mathbf{A} \in \mathcal{K}$. Finally, $\mathbf{A} \in \mathcal{K}$ iff $\mathbf{A} \models \Psi$ iff $\mathbf{A} \models \tilde{\Psi}$ \square

Let $\mathcal{V}_1, \mathcal{V}_2$ be subvarieties of RL axiomatized by E_1, E_2 , respectively, where E_1, E_2 have no variables in common.

The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the set of PUFs

$$\Psi = \{\forall \bar{x} (1 \leq r_1 \text{ or } 1 \leq r_2) \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2\}.$$

Theorem

$\mathcal{V}_1 \vee \mathcal{V}_2$ is axiomatized by

$$\tilde{\Psi} = \{\gamma_1(r_1) \vee \gamma_2(r_2) = 1 \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2, \gamma_i \in \Gamma\}$$

Representable RLs

Thm. The variety RRL generated by all linearly ordered residuated lattices is axiomatized by the identity $\lambda_z((x \vee y) \setminus x) \vee \rho_w((x \vee y) \setminus y) = 1$

Pf. A RL is a chain iff it satisfies $\forall x, y (x \leq y \text{ or } y \leq x)$, or

$$\forall x, y (1 \leq (x \vee y) \setminus x \text{ or } 1 \leq (x \vee y) \setminus y).$$

Thus, RRL is axiomatized by the identities

$$1 = \gamma_1((x \vee y) \setminus x) \vee \gamma_2((x \vee y) \setminus y); \gamma_1, \gamma_2 \in \Gamma \quad (\Gamma)$$

So, RRL satisfies the identity

$$\lambda_z((x \vee y) \setminus x) \vee \rho_w((x \vee y) \setminus y) = 1. \quad (\lambda, \rho)$$

Conversely, the variety axiomatized by this identity satisfies

$$x \vee y = 1 \Rightarrow \lambda_z(x) \vee y = 1 \quad x \vee y = 1 \Rightarrow x \vee \rho_w(y) = 1. \quad (\text{imp})$$

By repeated applications of (imp) on (λ, ρ) , we get (Γ) .

Lemma

Let \mathbf{A} be an *ibl*-groupoid and let $c \in B(\mathbf{A})$. Then $x \wedge c = xc = cx$ for all $x \in A$, and the Boolean center is a Boolean sublattice of central idempotent elements

Proof.

Suppose A is an *ibl*-groupoid and $c \wedge d = 0, c \vee d = 1$. By integrality

$$xc \leq x \wedge c = (x \wedge c)(c \vee d) = (x \wedge c)c \vee (x \wedge c)d \leq xc \vee 0 = xc,$$

and similarly $cx \leq c \wedge x \leq cx$.

Suppose we also have $a \wedge b = 0, a \vee b = 1$. To see that $B(\mathbf{A})$ is a sublattice of \mathbf{A} , it suffices to show that $a \vee c$ and $b \wedge d$ are complements:

$$(a \vee c) \wedge (b \wedge d) = (a \vee c)bd = abd \vee cbd = 0 \text{ and}$$

$$(a \vee c) \vee (b \wedge d) = a \vee c \vee bd = a \vee c \vee bc \vee bd = a \vee c \vee b(c \vee d) = a \vee c \vee b = 1.$$

Now $B(\mathbf{A})$ is complemented by definition, and it is a distributive lattice since \cdot distributes over \vee , hence it is a Boolean lattice. \square

Direct decompositions of FL_w -algebras

We now consider FL_w -algebras, i.e. residuated lattices with a 0 and 1 as the bottom and top element

Some of the results make no use of associativity, so they hold for integral bounded (residuated) groupoids (*ib(r)*-groupoids for short)

An element c in a FL_w -algebra \mathbf{A} is *complemented* if there exists $c' \in A$ such that $c \wedge c' = 0$ and $c \vee c' = 1$.

The *Boolean center* of \mathbf{A} is the set $B(\mathbf{A})$ of all complemented elements.

The *direct decomposition* results given here generalize similar results for MV-algebras [Cignoli, D'Ottaviano and Mundici 2000] and BL-algebras [Di Nola, Georgescu and Leustan 2000]

The first lemma is essentially due to [Birkhoff 1967].

The boolean center is closed under residuals

Lemma

If \mathbf{A} is a residuated *ibl*-groupoid then $B(\mathbf{A})$ is also closed under the residuals, the complement of c is $-c = 0/c = c \setminus 0$ and $c \setminus x = x/c = -c \vee x$ for all $c \in B(\mathbf{A})$ and $x \in A$.

Proof.

For complements c, d and any $x \in A$ we have

$$c \setminus x = (c \vee d)(c \setminus x) = c(c \setminus x) \vee d(c \setminus x) \leq x \vee d$$

On the other hand $c(x \vee d) = cx \vee cd \leq x$ implies $x \vee d \leq c \setminus x$.

Hence $c \setminus x = d \vee x$, and for $x = 0$ we obtain $-c = c \setminus 0 = d$

Therefore $c \setminus x = -c \vee x$ for all $x \in A$

The results for $/$ follow similarly. \square

For an $ib(r)\ell$ -groupoid \mathbf{A} and an element $c \in B(\mathbf{A})$, define

the *relativized subalgebra* $\mathbf{A}c = \downarrow c$ with unit $1^{\mathbf{A}c} = c$, operations \wedge, \vee, \cdot , restricted from \mathbf{A} ,

and $a \setminus b = (a \setminus^{\mathbf{A}} b) \wedge c$, $a / b = (a /^{\mathbf{A}} b) \wedge c$ for all $a, b \in \downarrow c$.

Lemma

For any $ib(r)\ell$ -groupoid \mathbf{A} and any $c \in B(\mathbf{A})$, the relativized subalgebra $\mathbf{A}c$ is an $ib(r)\ell$ -groupoid

Proof.

By the first lemma, $x \wedge c = xc = cx$, so $\mathbf{A}c$ has c as a unit and is closed under \wedge, \vee, \cdot , hence it is an $ib\ell$ -groupoid.

If \mathbf{A} has residuals then for all $a, b, x \in \mathbf{A}c$ we have

$$ax \leq b \quad \text{iff} \quad x \leq^{\mathbf{A}} a \setminus^{\mathbf{A}} b \text{ and } x \leq^{\mathbf{A}} c,$$

whence $a \setminus b = (a \setminus^{\mathbf{A}} b) \wedge c$, and similarly $a / b = (a /^{\mathbf{A}} b) \wedge c$. □

Complements in FL_w give direct decompositions

Theorem

If \mathbf{A} is an FL_w -algebra and if $c, d \in B(\mathbf{A})$ are complements then

$$\mathbf{A} \cong \mathbf{A}c \times \mathbf{A}d$$

Proof.

Consider the map $h : \mathbf{A} \rightarrow \mathbf{A}c \times \mathbf{A}d$ defined by $h(a) = (ac, ad)$.

The preceding lemma shows that h is a homomorphism, and

h has an inverse given by $(x, y) \mapsto x \vee y$

since $ac \vee ad = a(c \vee d) = a1 = a$ and

for $x \leq c, y \leq d$ we have

$$((x \vee y)c, (x \vee y)d) = (xc \vee yc, xd \vee yd) = (x, y)$$

□

Lemma

If \mathbf{A} is an FL_w -algebra and $c \in B(\mathbf{A})$ then the map $f : \mathbf{A} \rightarrow \mathbf{A}c$ given by

$$f(a) = ac \quad \text{is a surjective homomorphism,}$$

hence $\mathbf{A}c$ satisfies all identities that hold in \mathbf{A} .

Proof.

$$f(1) = 1c = 1^{\mathbf{A}c}, \quad (a \wedge b)c = a \wedge b \wedge c = ac \wedge bc \text{ and}$$

$$(a \vee b)c = ac \vee bc \quad \text{hence } f \text{ preserves } \wedge, \vee.$$

If \cdot is associative then $(ab)c = abcc = (ac)(bc)$.

In any residuated lattice $x \setminus y \leq xz \setminus yz$, hence $f(a \setminus^{\mathbf{A}} b) \leq f(a) \setminus f(b)$.

For the opposite inequality, we have $ac(ac \setminus bc) \leq bc \leq b$

and therefore $c(ac \setminus bc) \leq a \setminus b$. □

Direct decompositions imply complements

Conversely, any direct decomposition of an $ib(r)\ell$ -groupoid is obtained in this way, since the elements $(0, 1), (1, 0)$ are complements.

Corollary

A FL_w -algebra is directly indecomposable iff its Boolean center contains only the elements $\{0, 1\}$.

The structure of directly indecomposables can be further analyzed by using *subdirect decompositions*

However, we now consider a *poset product* that can even be used to decompose subdirectly irreducible algebras

Poset products

The *poset product* uses a partial order on the index set to define a subset of the direct product.

Specifically, let $\mathbf{X} = (X, \leq)$ be a poset, and assume $\{\mathbf{A}_i \mid i \in X\}$ is a family of algebras that have two constant operations denoted 0, 1.

The poset product of $\{A_i \mid i \in X\}$ is

$$\prod_{\mathbf{X}} A_i = \{f \in \prod_{i \in X} A_i \mid f(i) = 0 \text{ or } f(j) = 1 \text{ for all } i < j \text{ in } X\}$$

If \mathbf{X} is an *antichain* then the poset product is the same as the direct product

If \mathbf{X} is a *chain* and the A_i are ordered, then the poset product is the (amalgamated) ordinal sum of the A_i

We now generalize the poset sum decomposition result of [J., Montagna 2006] from finite GBL-algebras to certain FL_w -algebras

Theorem

Consider a FL_w -algebra \mathbf{A} with a finite subalgebra \mathbf{C} such that $C \subseteq I_{\mathbf{A}}$, and let \mathbf{X} be the dual of the partially ordered set of completely join irreducible elements of \mathbf{C} .

For $c \in X$, let c_* denote the unique lower cover of c in \mathbf{C} .

If $\mathbf{A}c = \downarrow c_* \oplus [c_*, c]$ for all $c \in X$ then $\mathbf{A} \cong \prod_{\mathbf{X}} [c_*, c]$.

The condition $\mathbf{A}c = \downarrow c_* \oplus [c_*, c]$ is actually satisfied for many GBL-algebras

For an ℓ -groupoid \mathbf{A} define $I_{\mathbf{A}} = \{c \in A \mid c \wedge x = cx = xc \text{ for all } x \in A\}$

Note that \wedge distributes over \vee in $I_{\mathbf{A}}$, but $I_{\mathbf{A}}$ need not be a subalgebra of \mathbf{A}

A *GBL-algebra* is a residuated lattice that satisfies

$$x \leq y \Rightarrow x = (x/y)y = y(y \setminus x)$$

[J., Montagna 2006] prove that for bounded GBL-algebras, $I_{\mathbf{A}}$ is a subalgebra, hence a Heyting algebra contained in \mathbf{A} ,

and $B(\mathbf{A})$ is the subalgebra of complemented elements of $I_{\mathbf{A}}$.

For MV-algebras $I_{\mathbf{A}} = B(\mathbf{A})$

Lemma

Let \mathbf{A} be a FL_w -algebra and let $a, b \in I_{\mathbf{A}}$ with $a \leq b$.

Then the interval $[a, b] = \{x \in A \mid a \leq x \leq b\}$ is a FL_w -algebra, with $0 = a, 1 = b$,

\wedge, \vee, \cdot inherited from \mathbf{A} , and $x \setminus y = (x \setminus^{\mathbf{A}} y) \wedge b, x/y = (x /^{\mathbf{A}} y) \wedge b$.

If \mathbf{A} is a GBL-algebra, then so is $[a, b]$.

A GBL-algebra is *normal* if every filter is a normal filter

Theorem (J., Montagna)

A Blok-Ferreirim decomposition for GBL-algebras: Every subdirectly irreducible normal integral GBL-algebra decomposes as the ordinal sum of an integral GBL-algebra and a linearly ordered integral GMV-algebra.

A residuated lattice is *n-potent* if it satisfies $x^{n+1} = x^n$

[J., Montagna] prove that any n -potent GBL-algebra is commutative, hence normal, so e.g. any finite GBL-algebra is commutative

Corollary

Suppose \mathbf{A} is an integral normal GBL-algebra such that $I_{\mathbf{A}}$ is finite

Then \mathbf{A} is isomorphic to a poset product of GMV-algebras

Conclusion

Residuated lattices have **many motivating examples** from algebra and logic

Residuation has powerful consequences for the generation of congruences

The structure theory of residuated lattices is still developing

Positive universal formulas can be translated to equations

For certain varieties (e.g. GBL) the structure theory is even **better behaved**

There are **many further interesting results** to be discovered

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