Semantics: Residuated Frames

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joint work with

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Outline

Part I

- Universal Algebra
- Examples of residuated lattices
- Congruences and normal filters
- The lattice of subvarieties
- Varieties generated by positive universal classes
- Direct decompositions and poset products

Part II

- Residuated Frames
- Decidability
- Poset products of residuated lattices

Part II: Frames of residuated lattices

We now consider semantics for residuated lattices

Kripke frames for modal logics are a useful tool for building counter models

They also led to many interesting notions such as frame completeness, canonical frames, and correspondence results

Aim: To present frames for arbitrary residuated lattices

and connect them with the proof theory of substructural logic

Galois connections and closure operators

For posets P, Q, maps $f : P \to Q$, $g : Q \to P$ are a *Galois connection* if

$$y \le f(x) \iff x \le g(y), \quad \text{for all } x \in P,\ y \in Q$$

A map $c : P \to P$ is a *closure operator* if $x \le y$ implies $c(x) \le c(y)$, $x \le c(x)$ and $c(c(x)) = c(x)$

Exercise: If f, g are a Galois connection then $c(x) = g(f(x))$ is a closure operator on P

A *lattice frame* is a structure $\mathbf{W} = (W, W', N)$ where W and W' are sets and N is a binary relation from W to W'

E.g. If $\mathbf{L}$ is a lattice, $\mathbf{W_L} = (L, L, \le)$ is a lattice frame

Let $J(\mathbf{L})$ be the set completely join irreducibles and $M(\mathbf{L})$ be the set completely meet irreducibles of $\mathbf{L}$. Then $\mathbf{L}_+ = (J(\mathbf{L}), M(\mathbf{L}), \le)$ is a lattice frame.

Lattice frames

For $X \subseteq W$ and $Y \subseteq W'$ we define the *polarities*

$$X^{\rhd} = \{b \in W' : x \, N \, b, \text{ for all } x \in X\}$$
$$Y^{\lhd} = \{a \in W : a \, N \, y, \text{ for all } y \in Y\}$$

Exercise: The maps $^{\rhd}: \mathcal{P}(W) \to \mathcal{P}(W')$ and $^{\lhd}: \mathcal{P}(W') \to \mathcal{P}(W)$ form a Galois connection

If $\gamma_N(X) = X^{\rhd\lhd}$ then $\gamma_N : \mathcal{P}(W) \to \mathcal{P}(W)$ is a closure operator

Lemma. If $\mathbf{L} = (L, \wedge, \vee)$ is a lattice and γ is a closure operator on $\mathbf{L}$, then $(\gamma[L], \wedge, \vee_\gamma)$ is a lattice where $x \vee_\gamma y = \gamma(x \vee y)$

Corollary. If $\mathbf{W}$ is a lattice frame then the *Galois algebra* $\mathbf{W}^+ = (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N})$ is a complete lattice

If $\mathbf{L}$ is a lattice, $\mathbf{W}_{\mathbf{L}}^+$ is the Dedekind-MacNeille completion of $\mathbf{L}$ and $x \mapsto \{x\}^{\lhd}$ is an embedding

Residuated frames

A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/)$ where W and W' are sets, $N \subseteq W \times W'$, $\circ \subseteq W^3$, $\backslash\!\!\backslash \subseteq W \times W' \times W$ and $/\!\!/ \subseteq W' \times W \times W$ such that for all $x, y \in W$, $w \in W'$

$$(x \circ y) \, N \, w \;\Leftrightarrow\; y \, N \, (x \backslash\!\!\backslash w) \;\Leftrightarrow\; x \, N \, (w /\!\!/ y)$$

Here $x \circ y = \{z \mid (x, y, z) \in \circ\}$ and similarly for $\backslash\!\!\backslash$, $/\!\!/$

We also use $X \, N \, y$ to abbreviate $x \, N \, y$ for all $x \in X$ and likewise for $x \, N \, Y$

A ternary relation structure $\mathbf{W} = (W, \circ)$ is said to be *associative* if it satisfies $(x \circ y) \circ z = x \circ (y \circ z)$, i.e., if it satisfies the following equivalence

$$\exists u[(x, y, u) \in \circ \text{ and } (u, z, w) \in \circ] \iff \exists v[(x, v, w) \in \circ \text{ and } (y, z, v) \in \circ]$$

It is said to *have a unit* $E \subseteq W$ if $x \circ E = \{x\} = E \circ x$, i.e., if

$$\exists e \in E[(x, e, y) \in \circ] \iff x = y \iff \exists e \in E[(e, x, y) \in \circ]$$

Nuclei

A *nucleus* γ on a residuated lattice $\mathbf{L}$ is a closure operator on L such that $\gamma(x)\gamma(y) \le \gamma(xy)$ (or $\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$).

Theorem. Given a RL $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ and a nucleus on $\mathbf{L}$, the algebra $\mathbf{L}_\gamma = (L_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$, is a residuated lattice, where $x \cdot_\gamma y = \gamma(x \cdot y)$, $x \vee_\gamma y = \gamma(x \vee y)$.

Theorem. For a frame $\mathbf{W}$, γ_N is a nucleus on $(\mathcal{P}(W), \cap, \cup, \circ, \backslash, /, \{1\})$

Corollary. If $\mathbf{W}$ is a residuated frame then the *Galois algebra* $\mathbf{W}^+ = (\mathcal{P}(W), \cap, \cup, \circ, \backslash, /, 1)_{\gamma_N}$ is a complete residuated lattice. Moreover, for $\mathbf{W}_{\mathbf{L}}$, $x \mapsto \{x\}^{\lhd}$ is an embedding.

If $\mathbf{L}$ is a RL, $\mathbf{W}_{\mathbf{L}} = (L, L, \le, \cdot, \backslash, /)$ is a residuated frame.

Frames of complete perfect lattices

A lattice $\mathbf{L}$ is *perfect* if every element is a join of elements of $J(\mathbf{L})$ and a meet of elements of $M(\mathbf{L})$

E.g. a Boolean algebra is perfect if and only if it is atomic

For a perfect residuated lattice $\mathbf{A}$, let $\mathbf{A}_+ = (J(\mathbf{A}), M(\mathbf{A}), \le, \circ, \backslash\!\!\backslash, /\!\!/, E)$ where $x \circ y = \{z \in J(\mathbf{A}) \mid z \le xy\}$ and $E = \{z \in J(\mathbf{A}) \mid z \le 1\}$

Theorem

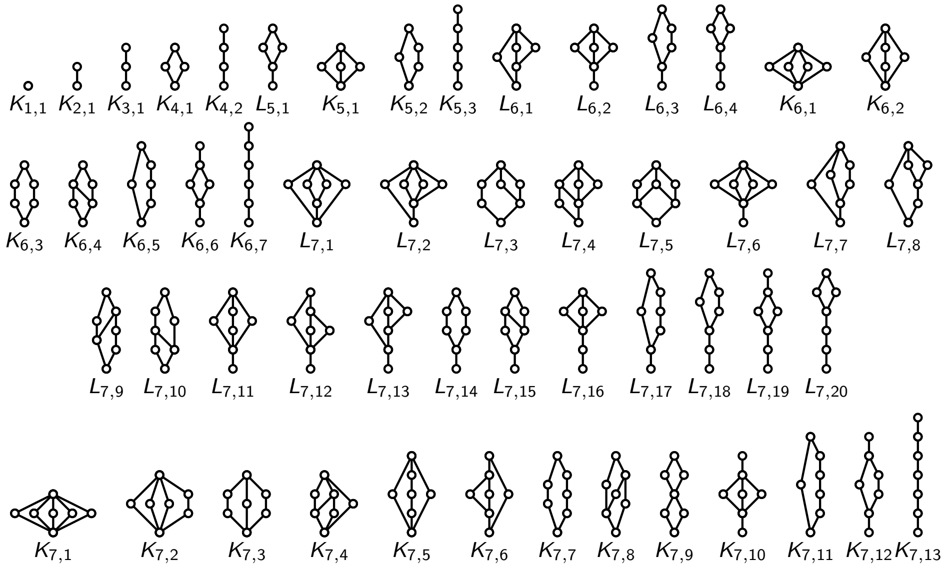
$\mathbf{A}_+$ is a residuated frame and if $\mathbf{A}$ is complete then $(\mathbf{A}_+)^+ \cong \mathbf{A}$

In particular, any finite lattice is complete and perfect

So for finite residuated lattices, residuated frames give a compact representation analogous to atom structures for relation algebras

All (dually-)nonisomorphic lattices with ≤ 7 elements

$K_{n,k}$ = selfdual lattice of size n $\qquad$ $L_{n,k}$ = nonselfdual lattices of size n



FL

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \text{L}\ell) \quad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \text{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \text{R})$$

$$\frac{y \circ a \circ z \Rightarrow c \quad y \circ b \circ z \Rightarrow c}{y \circ a \vee b \circ z \Rightarrow c}\ (\vee \text{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \text{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \text{R}r)$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \backslash b) \circ z \Rightarrow c}\ (\backslash \text{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \text{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c}\ (/\text{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\text{R})$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c}\ (\cdot \text{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \text{R})$$

$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

FL with context notation

$$\frac{x \Rightarrow a \quad u[a] \Rightarrow c}{u[x] \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{u[a] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \text{L}\ell) \quad \frac{u[b] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \text{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \text{R})$$

$$\frac{u[a] \Rightarrow c \quad u[b] \Rightarrow c}{u[a \vee b] \Rightarrow c}\ (\vee \text{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \text{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \text{R}r)$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[x \circ (a \backslash b)] \Rightarrow c}\ (\backslash \text{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \text{R})$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[(b/a) \circ x] \Rightarrow c}\ (/\text{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\text{R})$$

$$\frac{u[a \circ b] \Rightarrow c}{u[a \cdot b] \Rightarrow c}\ (\cdot \text{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \text{R})$$

$$\frac{u[\varepsilon] \Rightarrow a}{u[1] \Rightarrow a}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

Basic substructural logics

If the sequent s is provable in **FL** from the set of *sequents* S, we write $S \vdash_{\mathbf{FL}} s$.

$$\frac{u[x \circ y] \ \Rightarrow\ c}{u[y \circ x] \ \Rightarrow\ c}\ (e) \qquad (\text{exchange}) \qquad xy \leq yx$$

$$\frac{u[x \circ x] \ \Rightarrow\ c}{u[x] \ \Rightarrow\ c}\ (c) \qquad (\text{contraction}) \qquad x \leq x^2$$

$$\frac{u[\varepsilon] \ \Rightarrow\ c}{u[x] \ \Rightarrow\ c}\ (i) \qquad (\text{integrality}) \qquad x \leq 1$$

We write $\mathbf{FL_{ec}}$ for $\mathbf{FL} + (e) + (c)$.

Examples of frames (FL)

Consider the Gentzen system **FL** (full Lambek calculus).

We define the frame $\mathbf{W_{FL}}$, where

- $(W, \circ, \varepsilon)$ to be the free monoid over the set Fm of all formulas
- $W' = S_W \times Fm$, where S_W is the set of all *unary linear polynomials* $u[x] = y{\circ}x{\circ}z$ of W, and
- $x\ N\ (u, a)$ iff $\vdash_{\mathbf{FL}} u[x] \ \Rightarrow\ a$.

For $(u, a) /\!\!/ x = \{(u[\_ \circ x], a)\}$ and $x \backslash\!\!\backslash (u, a) = \{(u[x \circ \_], a)\}$, we have

$$\begin{aligned} x \circ y N(u, a) \quad & \text{iff } \vdash_{\mathbf{FL}} u[x \circ y] \ \Rightarrow\ a \\ & \text{iff } \vdash_{\mathbf{FL}} u[x{\circ}y] \ \Rightarrow\ a \\ & \text{iff } xN(u[\_ \circ y], a) \\ & \text{iff } yN(u[x \circ \_], a). \end{aligned}$$

Examples of frames (FEP)

Let **A** be a residuated lattice and **B** a partial subalgebra of **A**.

We define the frame $\mathbf{W_{A,B}}$, where

- $(W, \cdot, 1)$ to be the submonoid of **A** generated by B,
- $W' = S_B \times B$, where S_W is the set of all *unary linear polynomials* $u[x] = y{\circ}x{\circ}z$ of $(W, \cdot, 1)$, and
- $x\ N\ (u, b)$ by $u[x] \leq_{\mathbf{A}} b$.

For $(u, a) /\!\!/ x = \{(u[\_ \cdot x], a)\}$ and $x \backslash\!\!\backslash (u, a) = \{(u[x \cdot \_], a)\}$, we have

$$\begin{aligned} x \cdot y N(u, a) \quad & \text{iff } u[x \cdot y] \leq a \\ & \text{iff } xN(u[\_ \cdot y], a) \\ & \text{iff } yN(u[x \cdot \_], a). \end{aligned}$$

GN

$$\frac{xNa \quad aNz}{xNz}\ (\mathrm{CUT}) \qquad \frac{}{aNa}\ (\mathrm{Id})$$

$$\frac{xNa \quad bNz}{x \circ (a\backslash b)Nz}\ (\backslash\mathrm{L}) \qquad \frac{a \circ xNb}{xNa\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{xNa \quad bNz}{(b/a) \circ xNz}\ (/\mathrm{L}) \qquad \frac{x \circ aNb}{xNb/a}\ (/\mathrm{R})$$

$$\frac{a \circ bNz}{a \cdot bNz}\ (\cdot\mathrm{L}) \qquad \frac{xNa \quad yNb}{x \circ yNa \cdot b}\ (\cdot\mathrm{R})$$

$$\frac{aNz}{a \wedge bNz}\ (\wedge\mathrm{L}\ell) \qquad \frac{bNz}{a \wedge bNz}\ (\wedge\mathrm{L}r) \qquad \frac{xNa \quad xNb}{xNa \wedge b}\ (\wedge\mathrm{R})$$

$$\frac{aNz \quad bNz}{a \vee bNz}\ (\vee\mathrm{L}) \qquad \frac{xNa}{xNa \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{xNb}{xNa \vee b}\ (\vee\mathrm{R}r)$$

$$\frac{\varepsilon Nz}{1Nz}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon N1}\ (1\mathrm{R})$$

Gentzen frames

The following properties hold for $\mathbf{W_L}$, $\mathbf{W_{FL}}$ and $\mathbf{W_{A,B}}$:

1. **W** is a residuated frame
2. **B** is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$
3. B generates $(W, \circ, \varepsilon)$ (as a monoid)
4. W' contains a copy of B $(b \leftrightarrow (id, b))$
5. N satisfies **GN**, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

A *cut-free Gentzen frame* is not assumed to satisfy the (CUT)-rule.

Theorem. Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map $\{\}^{\lhd} : \mathbf{B} \to \mathbf{W}^{+},\ \ b \mapsto \{b\}^{\lhd}$ is a (partial) homomorphism.

(Namely, if $a, b \in B$ and $a \bullet b \in B$ ($\bullet$ is a connective) then $\{a \bullet_{\mathbf{B}} b\}^{\lhd} = \{a\}^{\lhd} \bullet_{\mathbf{W}^+} \{b\}^{\lhd}$).

Proof

Key Lemma. Let $(\mathbf{W}, \mathbf{B})$ be a Gentzen frame. For all $a, b \in B$, $X, Y \in \mathbf{W}^+$ and for every connective $\bullet$, if $a \bullet b \in B$, $a \in X \subseteq \{a\}^{\lhd}$ and $b \in Y \subseteq \{b\}^{\lhd}$, then

1. $a \bullet_{\mathbf{B}} b \in X \bullet_{\mathbf{W}^+} Y \subseteq \{a \bullet_{\mathbf{B}} b\}^{\lhd}$ $(1_{\mathbf{B}} \in 1_{\mathbf{W}^+} \subseteq \{1_{\mathbf{B}}\}^{\lhd})$
2. In particular, $a \bullet_{\mathbf{B}} b \in \{a\}^{\lhd} \bullet_{\mathbf{W}^+} \{b\}^{\lhd} \subseteq \{a \bullet_{\mathbf{B}} b\}^{\lhd}$.
3. Furthermore, because of (CUT), we have equality.

Proof Let $\bullet = \vee$. If $x \in X$, then $x \in \{a\}^{\lhd}$; so xNa and $xNa \vee b$, by $(\vee R\ell)$; hence $x \in \{a \vee b\}^{\lhd}$ and $X \subseteq \{a \vee b\}^{\lhd}$. Likewise $Y \subseteq \{a \vee b\}^{\lhd}$, so $X \cup Y \subseteq \{a \vee b\}^{\lhd}$ and $X \vee Y = \gamma(X \cup Y) \subseteq \{a \vee b\}^{\lhd}$.

On the other hand, let $X \vee Y \subseteq \{z\}^{\lhd}$, for some $z \in W$. Then, $a \in X \subseteq X \vee Y \subseteq \{z\}^{\lhd}$, so aNz. Similarly, bNz, so $a \vee bNz$ by $(\vee L)$, hence $a \vee b \in \{z\}^{\lhd}$. Thus, $a \vee b \in X \vee Y$.

We used that every closed set is an intersection of *basic closed sets* $\{z\}^{\lhd}$, for $z \in W$.

DM-completion

For a residuated lattice $\mathbf{L}$, we associated the Gentzen frame $(\mathbf{W_L}, \mathbf{L})$.

The underlying poset of $\mathbf{W}_{\mathbf{L}}^+$ is the *Dedekind-MacNeille completion* of the underlying poset reduct of $\mathbf{L}$.

Theorem. The map $x \mapsto x^{\lhd}$ is an embedding of $\mathbf{L}$ into $\mathbf{W}_{\mathbf{L}}^+$.

Completeness - Cut elimination

For every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, let $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{W}^+$ be the homomorphism that extends $\bar{f}(p) = \{f(p)\}^{\lhd}$ for any variable p

Corollary. If $(\mathbf{W}, \mathbf{B})$ is a cf Gentzen frame then for every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, we have $f(a) \in \bar{f}(a) \subseteq \{f(a)\}^{\lhd}$. With CUT $\bar{f}(a)=\{f(a)\}^{\lhd}$

We define $\mathbf{W_{FL}} \models x \Rightarrow c$ if $f(x)\ N\ f(c)$ for all $f : \mathbf{Fm} \to \mathbf{Fm}$

Theorem. If $\mathbf{W}_{\mathbf{FL}}^+ \models x^{\cdot} \leq c$, then $\mathbf{W_{FL}} \models x \Rightarrow c$.
Idea: For $f : \mathbf{Fm} \to \mathbf{B}$, $f(x) \in \bar{f}(x) \subseteq \bar{f}(c) \subseteq \{f(c)\}^{\lhd}$, so $f(x)\ N\ f(c)$.

Corollary. **FL** is complete with respect to $\mathbf{W}_{\mathbf{FL}}^+$.
Corollary. The algebra $\mathbf{W}_{\mathbf{FL}}^+$ generates RL.

The frame $\mathbf{W_{FL^f}}$ corresponds to cut-free **FL**.

Corollary (CE). **FL** and $\mathbf{FL^f}$ prove the same sequents.

Corollary. **FL** and the equational theory of RL are decidable.

Finite model property

For $\mathbf{W_{FL}}$, given $(x, z) \in W \times W'$ (if $z = (u, c)$, then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of $\mathbf{FL^f}$. Note that $(x, z)^{\uparrow}$ is finite.

The new frame $\mathbf{W}'$ associated with $N' = N \cup ((y, v)^{\uparrow})^c$ is residuated and Gentzen.

$(N')^c$ is finite, so has finite domain $Dom((N')^c)$ and codomain $Cod((N')^c)$

For every $z \notin Cod((N')^c)$, $\{z\}^{\lhd} = W$. So, $\{\{z\}^{\lhd} : z \in W\}$ is finite and a basis for $\gamma_{N'}$. So $\mathbf{W}'^+$ is finite.

Moreover, if $u(x) \Rightarrow c$ is not provable in **FL**, then it is not valid in $\mathbf{W}'^+$.

Corollary. The system **FL** has the finite model property.

Corollary. The variety **RL** is generated by its finite members.

FEP

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

The corresponding logic has the *strong finite model property*:

if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model, namely there is $\mathbf{D} \in \mathcal{K}$ and a homomorphism $f : \mathbf{Fm} \to \mathbf{D}$, such that $f(\phi) = 1$, for all $\phi \in \Phi$, but $f(\psi) \neq 1$.

For logics with finitely many axioms and rules, this implies that the deducibility relation is decidable

For a finitely axiomatized variety, FEP implies the decidability of the universal theory

FEP for integral RLs with $\{\vee, \cdot, 1\}$-equations

Blok and van Alten 2002 proved FEP for integral RLs, and extended it to residuated groupoids (2005)

Theorem. Every variety of integral RL's axiomatized by equations over $\{\vee, \cdot, 1\}$ has the FEP.

- $\mathbf{B}$ embeds in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ via $\{\_\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^+$
- $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite
- $\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$

Corollary. These varieties are generated as quasivarieties by their finite members.

Corollary. The corresponding logics have the *strong finite model property*

Finiteness

Idea for finiteness: Every element in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is an intersection of basic elements. So it suffices to prove that there are only finitely many such elements.

Replace the frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$ by one $\mathbf{W}^{\mathbf{M}}_{\mathbf{A},\mathbf{B}}$, where it is easier to work.

Let $\mathbf{M}$ be the free monoid with unit over the set B and $f : M \to W$ the extension of the identity map.

$$M \xrightarrow{f} W \overset{N}{—} W'$$

.

Equations 1

Idea: Express equations over $\{\vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: monoid terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 y \leq xy \vee yx$

$(x_1 \vee x_2)^2 y \leq (x_1 \vee x_2)y \vee y(x_1 \vee x_2)$

$x_1^2 y \vee x_1x_2y \vee x_2x_1y \vee x_2^2 y \leq x_1y \vee x_2y \vee yx_1 \vee yx_2$

$x_1x_2y \leq x_1y \vee x_2y \vee yx_1 \vee yx_2$

$$\frac{x_1y \leq v \quad x_2y \leq v \quad yx_1 \leq v \quad yx_2 \leq v}{x_1x_2y \leq v}$$

$$\frac{x_1 \circ y \mathrel{N} z \quad x_2 \circ y \mathrel{N} z \quad y \circ x_1 \mathrel{N} z \quad y \circ x_2 \mathrel{N} z}{x_1 \circ x_2 \circ y \mathrel{N} z}\ R(\varepsilon)$$

Equations 2

Theorem. If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies R(ε) iff $\mathbf{W}^+$ satisfies ε.

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Corollary. If an equation over $\{\vee, \cdot, 1\}$ is valid in $\mathbf{A}$, then it is also valid in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$, for every partial subalgebra $\mathbf{B}$ of $\mathbf{A}$.

Consequently, $\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$.

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \; (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *linear* if all variables in t_0 are distinct.

Theorem. Every system obtained from **FL** by adding linear rules has the cut elimination property.

A set of rules of the form $R(\varepsilon)$ is called *reducing* if there is a complexity measure that decreases with upward applications of the rules (and the rules of **FL**).

Theorem. Every system obtained from **FL** by adding linear reducing rules is decidable. The subvariety of residuated lattices axiomatized by the corresponding equations has decidable equational theory.

Applications

- Cut-elimination (CE) and finite model property (FMP) for **FL** and (cyclic) **InFL**. Generation by finite members for RL, InFL
- M. Kozak 2008 proved distributive **FL** has the FMP, and using our approach the same result holds for any extension of **DFL** with linear reducing structural rules
- The finite embeddability property (FEP) for integral RL with $\{\vee, \cdot, 1\}$-axioms
- The above extend to the non-associative case, also with the addition of suitable structural rules

(Un)decidability

Theorem. The quasiequational theory of RL is undecidable. (Because we can embed semigroups/monoids.) The same holds for commutative RL.

A lattice is *modular* if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$

Theorem. The equational theory of modular RL is undecidable. (Using the corresponding result for modular lattices by Freese 1980).

Theorem. The equational theory of commutative, distributive RL is decidable (Galatos Raftery 2004, from decidability of relevant logic RW by Brady 1990).

Word problem

A finitely presented algebra $\mathbf{A} = (X|R)$ (in a class $\mathcal{K}$) has a *solvable word problem* (WP) if there is an algorithm that, given any pair of words over X, decides if they are equal or not.

A class of algebras has *solvable WP* if all finitely presented algebras in it do.

For example, the varieties of semigroups, groups, ℓ-groups, modular lattices have unsolvable WP.

Theorem [Galatos 2002]: The variety CDRL of commutative, distributive residuated lattices has unsolvable WP.

Word problem for CDRL is unsolvable

Main idea: Embed semigroups, whose WP is unsolvable.

Residuated lattices have a semigroup operation $\cdot$, but commutative semigroups have a decidable WP.

Alternative approach: Come up with another term definable operation $\odot$ in commutative distributive residuated lattices that is associative and embeds all semigroups.

Technique: Coordinization in projective geometry and modular lattices, developed by J. von Neumann for continuous geometries, and applied by R. Freese to modular lattices, A. Urquhart to relevance logics, H. Andreak, S. Givant, I. Nemeti to symmetric relation algebras, and N. Galatos to CDRL

Undecidability of quasiequational theory

A *quasi-equation* is a formula of the form

$$(s_1 = t_1 \;\&\; s_2 = t_2 \;\&\; \cdots \;\&\; s_n = t_n) \Rightarrow s = t$$

The decidability of the *quasi-equational theory* states that there is an algorithm that decides all quasi-equations of the above form.

The equivalent logical notion is the decidability of the deducibility relation for formulas.

Corollary The *quasi-equational* theory of CDRL is undecidable.

Hence CDRL does not have the FEP, although we saw earlier that the equational theory is decidable.

Further results on decidability

C. Holland, S. H. McCleary 1979: ℓ-groups have decidable equation theory

A.M.W. Glass, Y. Gurevich 1983: ℓ-groups have undecidable word problem

N. G. Hisamiev 1966: abelian ℓ-groups have decidable universal theory, by V. Weispfenning 1986, in fact co-NP-complete, but by Y. Gurevich 1967, the first-order theory is hereditarily undecidable

MV-algebras have FEP because of a connections to linear programming

IGMV and GMV have decidable equational theory because of a connections to ℓ-groups (N. Galatos, C. Tsinakis 2004), but no FMP

P. Jipsen and F. Montagna 2006, 2008: GBL and IGBL do not have FMP but normal GBL-algebras have FEP

Poset products

The *poset product* uses a partial order on the index set to define a subset of the direct product.

Specifically, let $\mathbf{X} = (X, \leq)$ be a poset, and assume $\{\mathbf{A}_i \mid i \in X\}$ is a family of algebras that have two constant operations denoted $0, 1$.

The poset product of $\{A_i \mid i \in X\}$ is

$$\prod_{\mathbf{X}} A_i = \{f \in \prod_{i \in X} A_i \mid \quad f(i) = 0 \text{ or } f(j) = 1 \quad \text{for all } i < j \text{ in } X\}$$

If $\mathbf{X}$ is an *antichain* then the poset product is the same as the direct product

If $\mathbf{X}$ is a *chain* and the A_i are ordered, then the poset product is the (amalgamated) ordinal sum of the A_i

For an ℓ-groupoid $\mathbf{A}$ define $I_{\mathbf{A}} = \{c \in A \mid c \wedge x = cx = xc \text{ for all } x \in A\}$

Note that $\wedge$ distributes over $\vee$ in $I_{\mathbf{A}}$, but $I_{\mathbf{A}}$ need not be a subalgebra of $\mathbf{A}$

A *GBL-algebra* is a residuated lattice that satisfies
$x \leq y \Rightarrow x = (x/y)y = y(y\backslash x)$

[J., Montagna 2006] prove that for bounded GBL-algebras, $\mathbf{I}_{\mathbf{A}}$ is a subalgebra, hence a Heyting algebra contained in $\mathbf{A}$,

and $B(\mathbf{A})$ is the subalgebra of complemented elements of $\mathbf{I}_{\mathbf{A}}$.

For MV-algebras $\mathbf{I}_{\mathbf{A}} = B(\mathbf{A})$

Lemma

Let $\mathbf{A}$ be a FL_w-algebra and let $a, b \in I_{\mathbf{A}}$ with $a \leq b$.
Then the interval $[a, b] = \{x \in A \mid a \leq x \leq b\}$ is a FL_w-algebra, with $0 = a$, $1 = b$,
$\wedge$, $\vee$, $\cdot$ inherited from $\mathbf{A}$, and $x\backslash y = (x\backslash^{\mathbf{A}} y) \wedge b$, $x/y = (x/^{\mathbf{A}} y) \wedge b$.
If $\mathbf{A}$ is a GBL-algebra, then so is $[a, b]$.

We now generalize the poset sum decomposition result of [J., Montagna 2006] from finite GBL-algebras to certain FL_w-algebras

Theorem

Consider a FL_w-algebra $\mathbf{A}$ with a finite subalgebra $\mathbf{C}$ such that $C \subseteq I_{\mathbf{A}}$,

and let $\mathbf{X}$ be the dual of the partially ordered set of completely join irreducible elements of $\mathbf{C}$.

For $c \in X$, let c_* denote the unique lower cover of c in $\mathbf{C}$.

If $\mathbf{A}c = {\downarrow}c_* \oplus [c_*, c]$ for all $c \in X$ then $\mathbf{A} \cong \prod_{\mathbf{X}} [c_*, c]$.

The condition $\mathbf{A}c = {\downarrow}c_* \oplus [c_*, c]$ is actually satisfied for many GBL-algebras

A GBL-algebra is *normal* if every filter is a normal filter

Theorem (J., Montagna)

A Blok-Ferreirim decomposition for GBL-algebras: Every subdirectly irreducible normal integral GBL-algebra decomposes as the ordinal sum of an integral GBL-algebra and a linearly ordered integral GMV-algebra.

A residuated lattice is *n-potent* if it satisfies $x^{n+1} = x^n$

[J., Montagna] prove that any n-potent GBL-algebra is commutative, hence normal, so e.g. any finite GBL-algebra is commutative

Corollary

Suppose $\mathbf{A}$ is an integral normal GBL-algebra such that $I_{\mathbf{A}}$ is finite

Then $\mathbf{A}$ is isomorphic to a poset product of linearly ordered IGMV-algebras

Open Problems

Do cancellative residuated lattices have a decidable equational theory or a cut free Gentzen system?

Do (I)GBL-algebras have a decidable equational theory or a cut free Gentzen system?

Is provability in $\mathbf{FL}_c$ decidable, i.e. does the variety FL_c have a decidable equational theory? A cut-free Gentzen system is known

Develop a structure theory for infinite IGBL-algebras

Do commutative cancellative residuated lattices satisfy any lattice equations that do not hold in all lattices?

Investigate the structure of free residuated lattices. Even the 1-generated case is not well understood

Find practical decision procedures for deducibility in $\mathbf{FL}_{(e)w}$ or for deciding quasiequations in (C)IRL

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