# Computational Investigations of the Lattice of Lattice Varieties

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# Outline

What is a lattice?

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Future work

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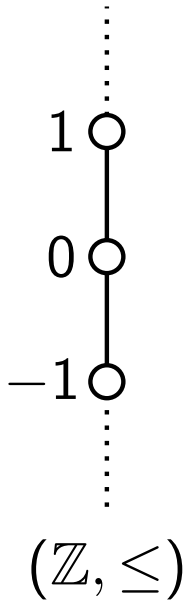


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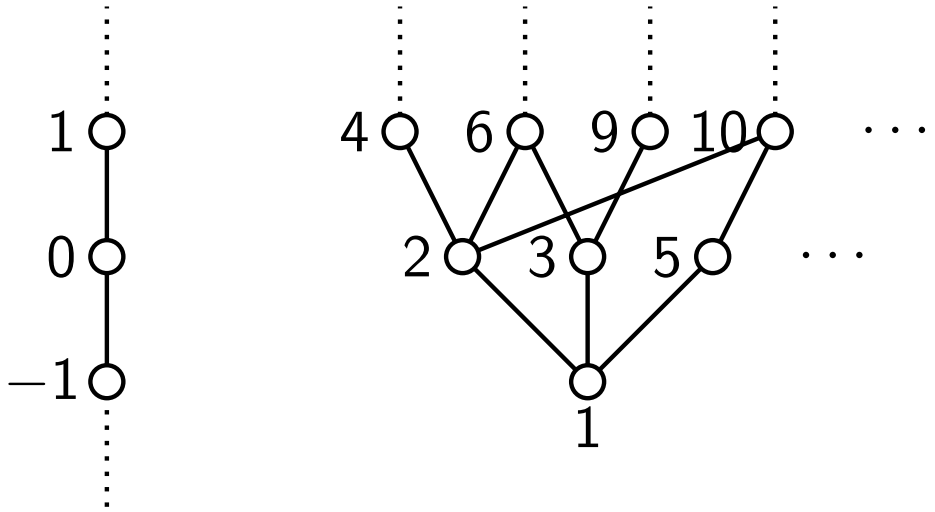

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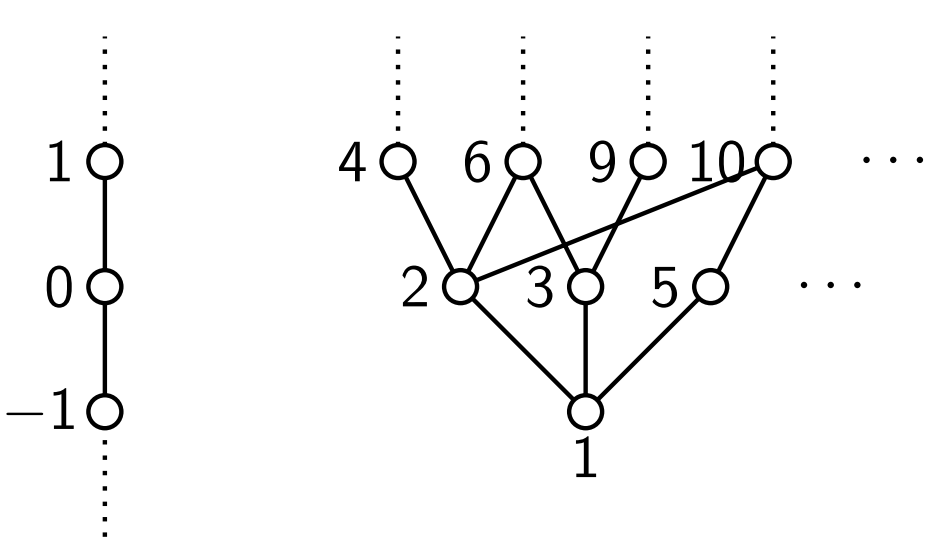


$(\mathbb{Z}, \leq)$ $(\mathbb{N}, |)$ $\mathcal{P}(\{1, 2, 3\})$

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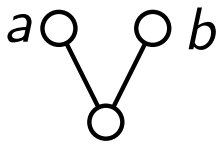


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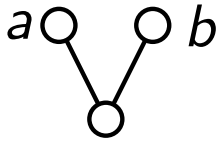


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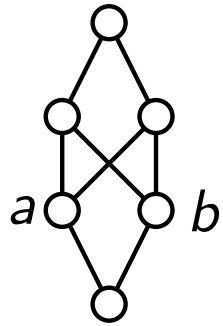


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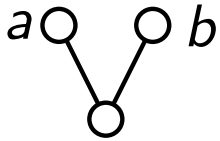


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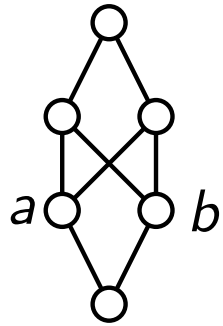


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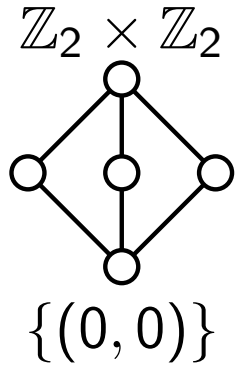


$M_3$=a subgroup lattice

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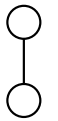
E.g. $\mathcal{D} = \mathsf{V}(\{$  $\})$

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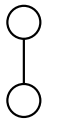
E.g. $\mathcal{D} = \mathrm{V}(\{$  $\})$

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Lattices of vector subspaces, lattices of normal subgroups, ... all modular

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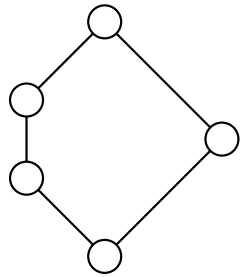

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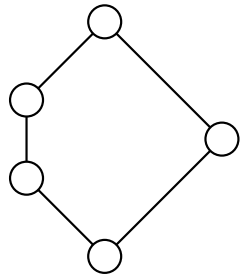


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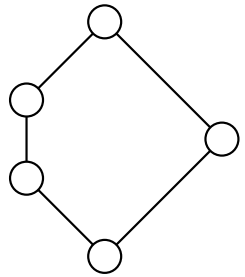


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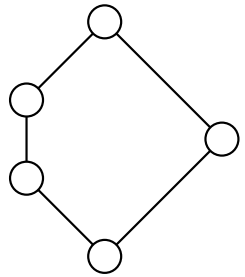


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Garrett Birkhoff [1935]: every nondistributive modular lattice contains $M_3$ as a sublattice

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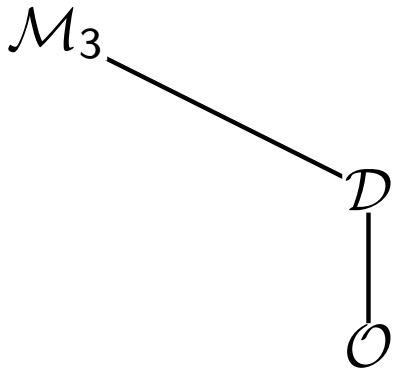


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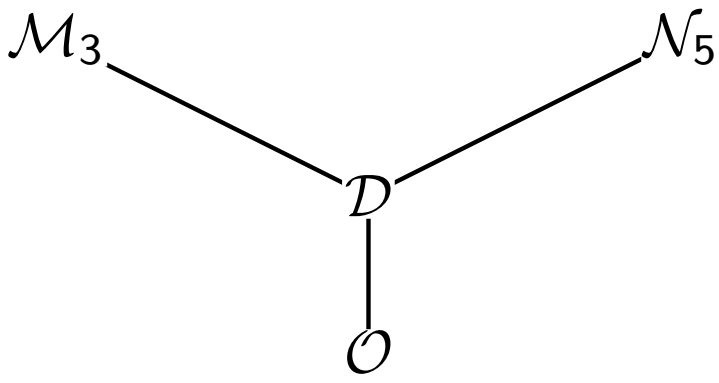


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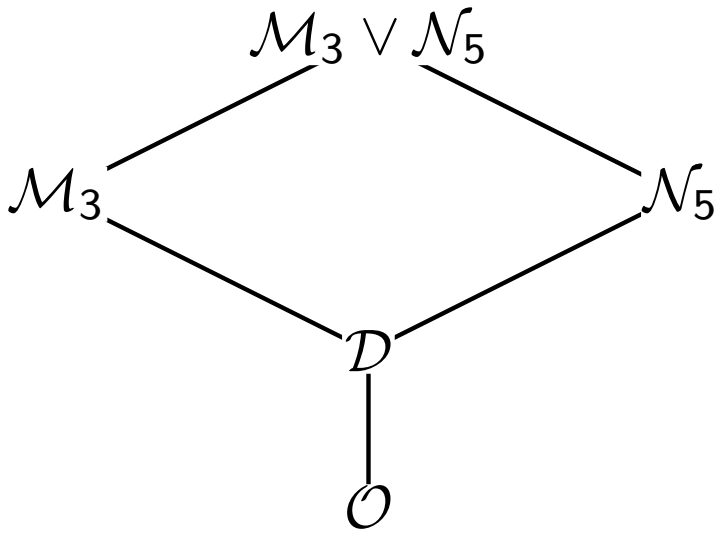


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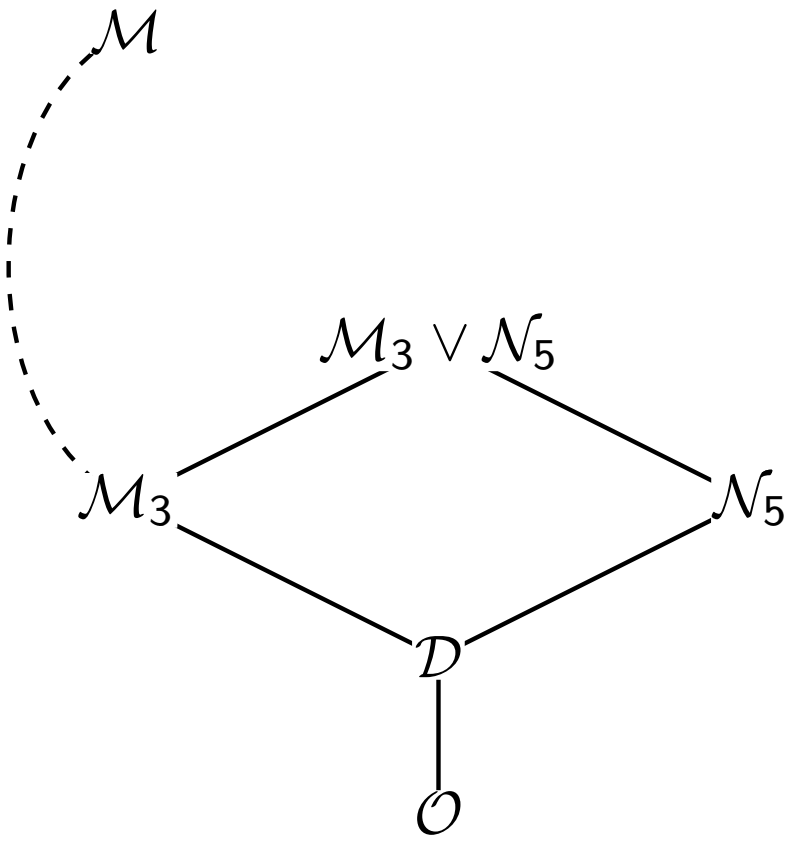


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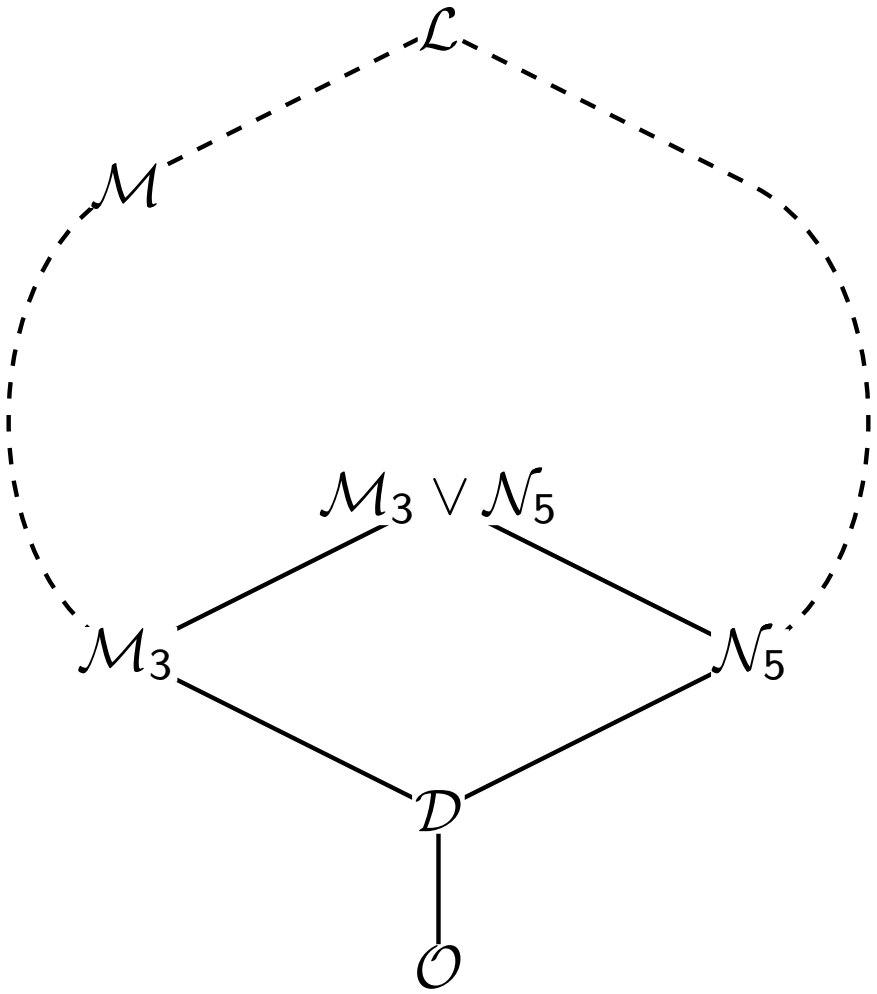


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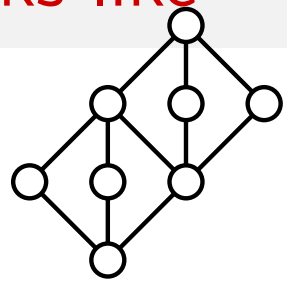

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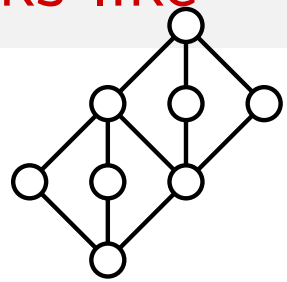

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Bjarni Jónsson [1968] removed the restriction that the variety be finitely generated by proving the following result:

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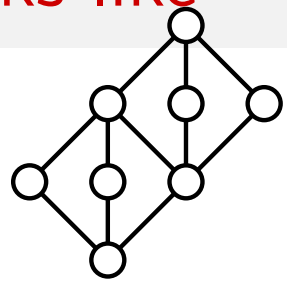

George Grätzer [1966]: any finitely generated modular variety strictly above $\mathcal{M}_3$ is above $\mathcal{M}_4$ or $\mathcal{M}_{3^2}$ (or both)

Bjarni Jónsson [1968] removed the restriction that the variety be finitely generated by proving the following result:

**Theorem**

*For a modular variety $\mathcal{V}$ the following conditions are equivalent:*

1. $M_{3^2} \notin \mathcal{V}$,
2. *every subdirectly irreducible member of $\mathcal{V}$ has length $\leq 2$,*
3. *the inequality $x \wedge (y \vee (z \wedge w)) \wedge (z \vee w) \leq y \vee (x \wedge z) \vee (x \wedge w)$ holds in $\mathcal{V}$.*

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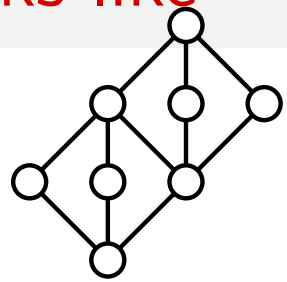

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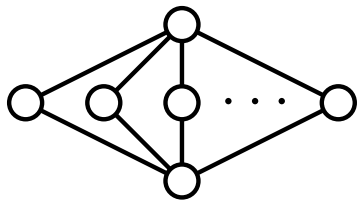
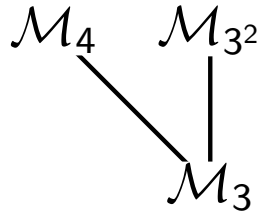
The only lattices of length 2 are $M_n$



($n+2$ elements)

$\mathcal{M}_3$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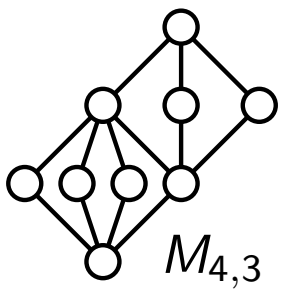
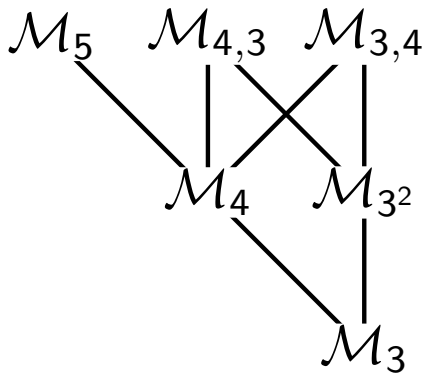
$\mathcal{M}_3$


$M_{4,3}$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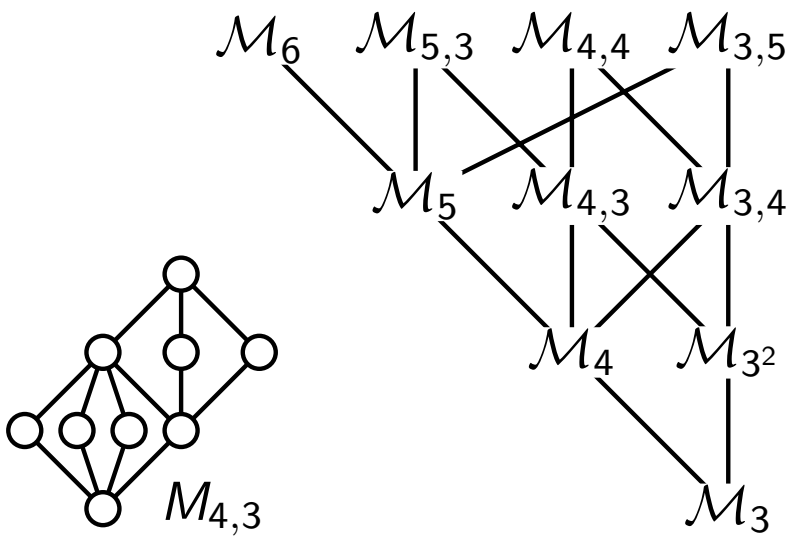
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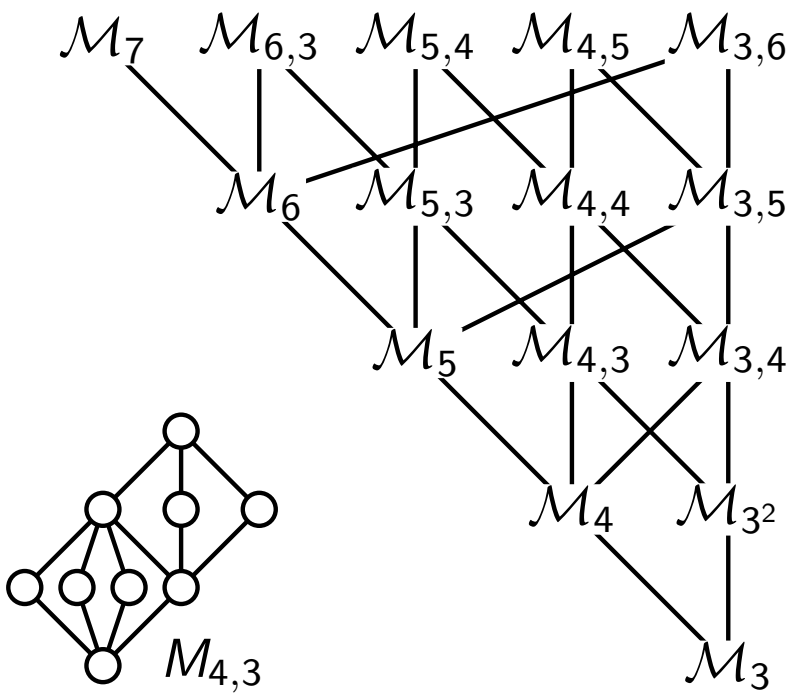
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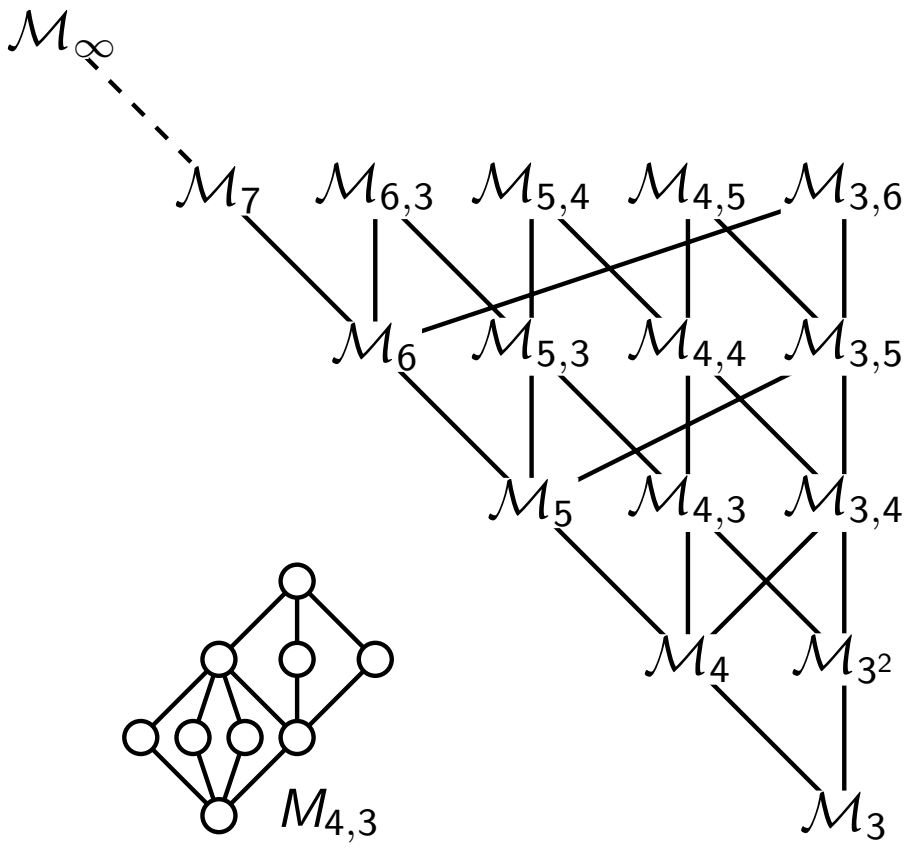
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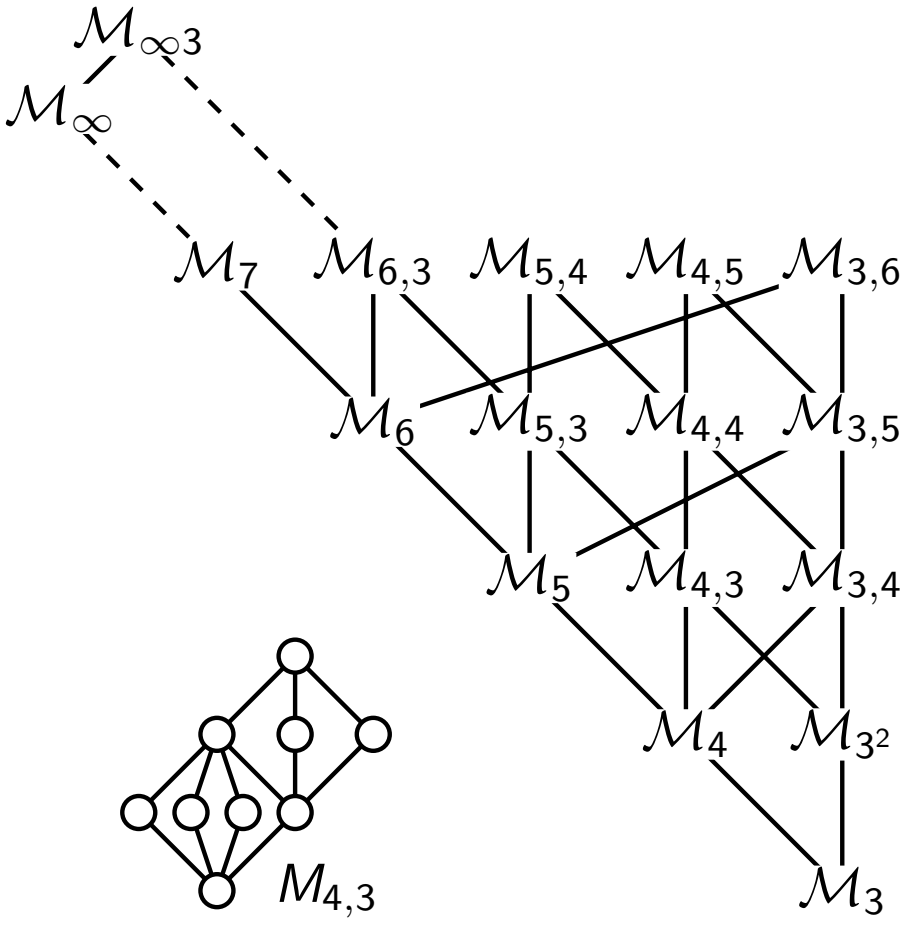
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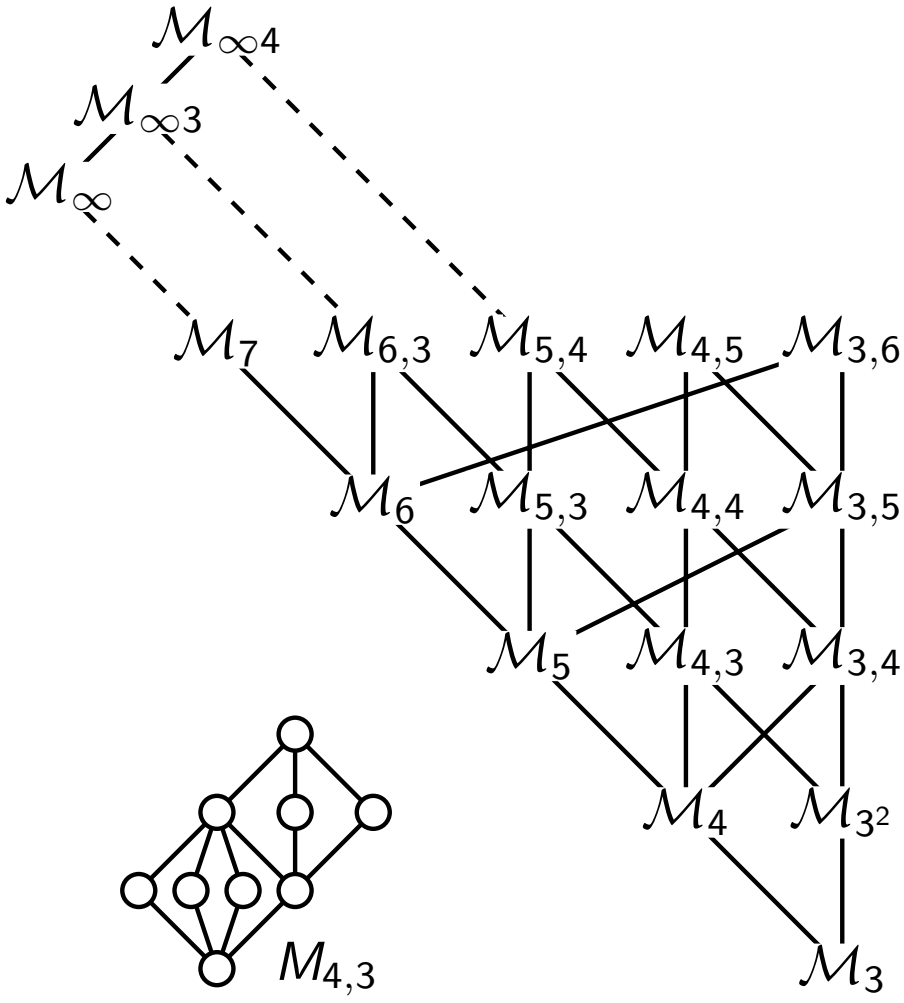
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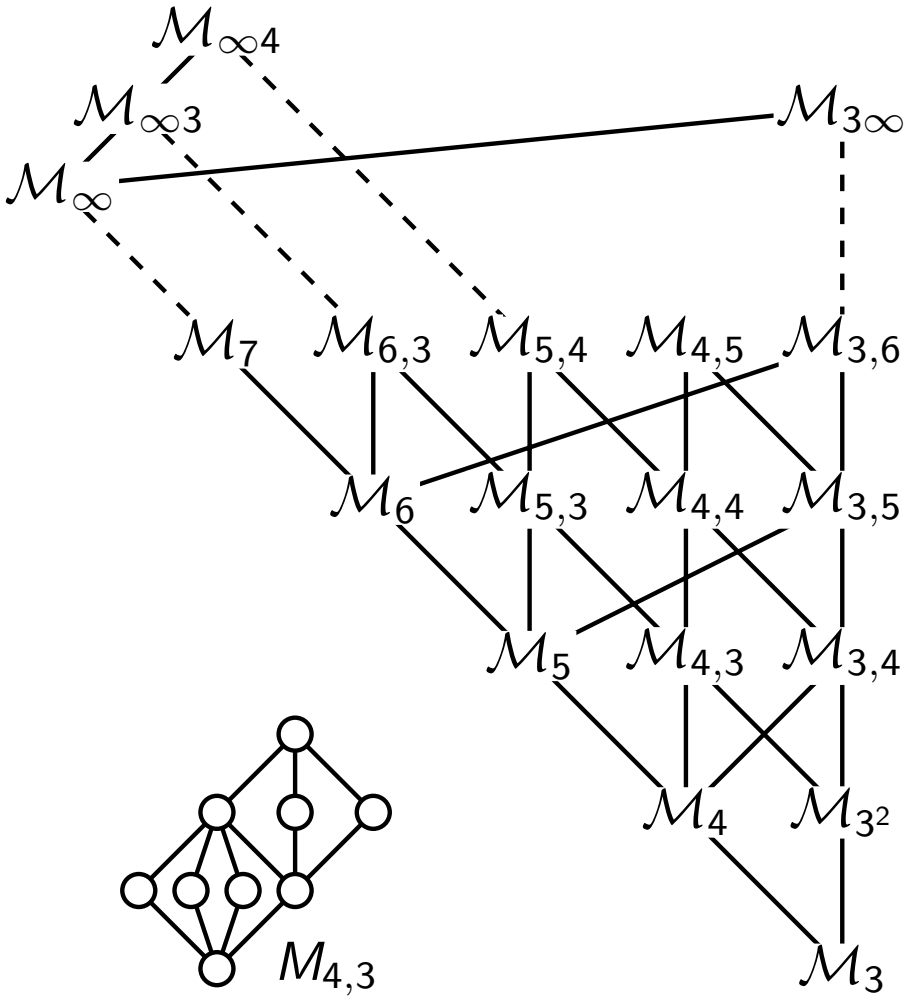
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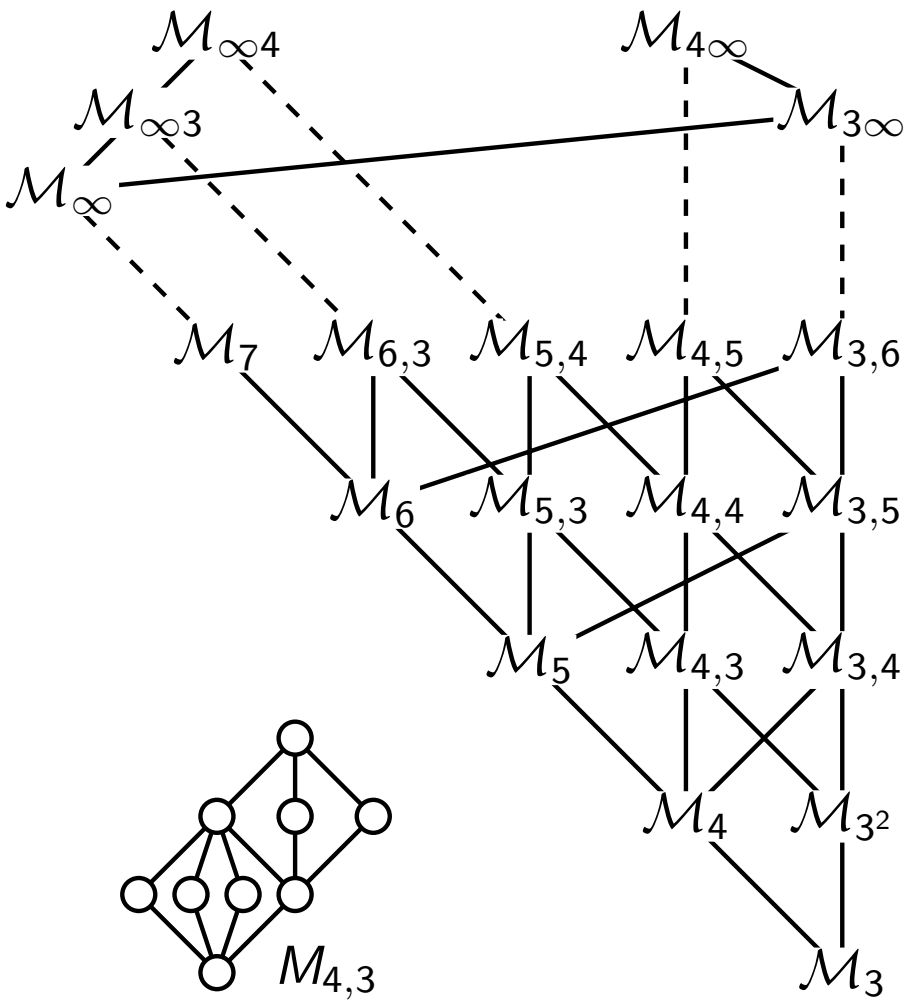
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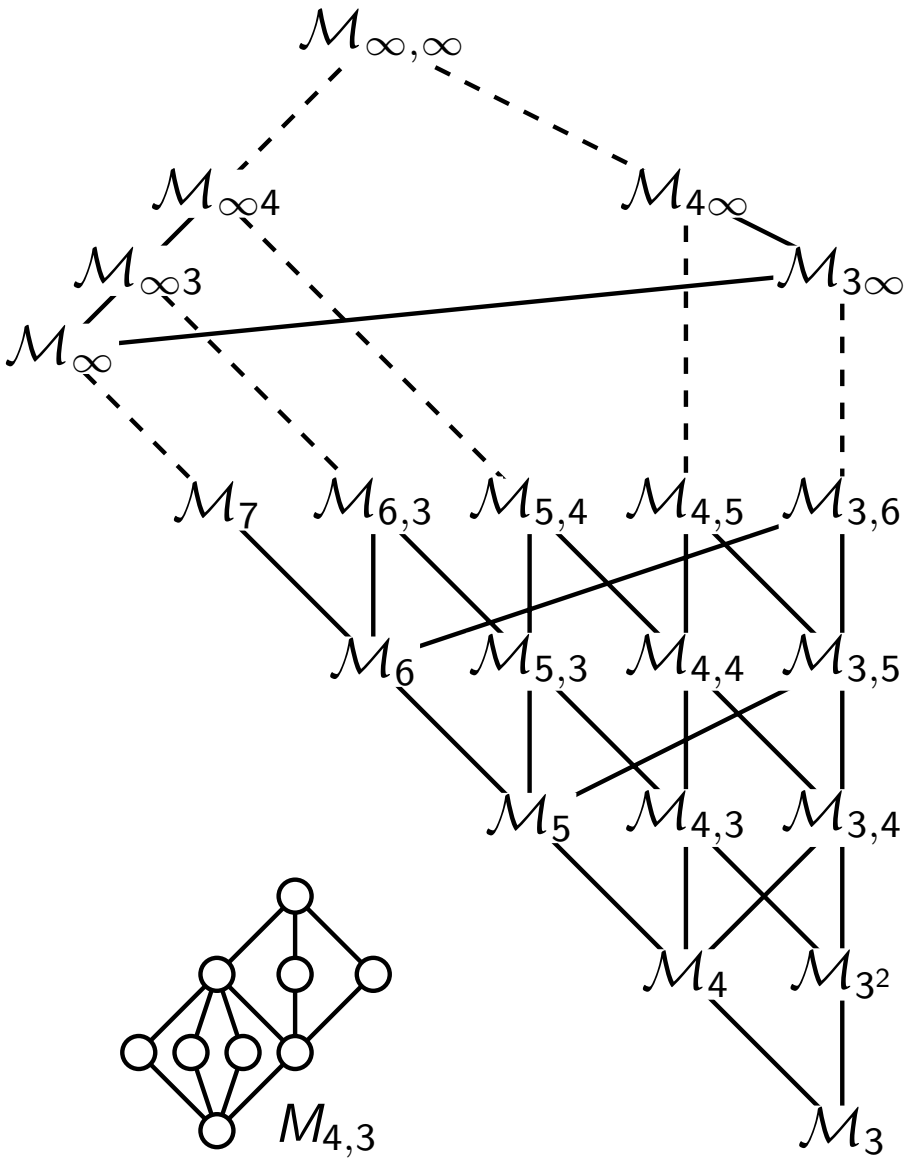
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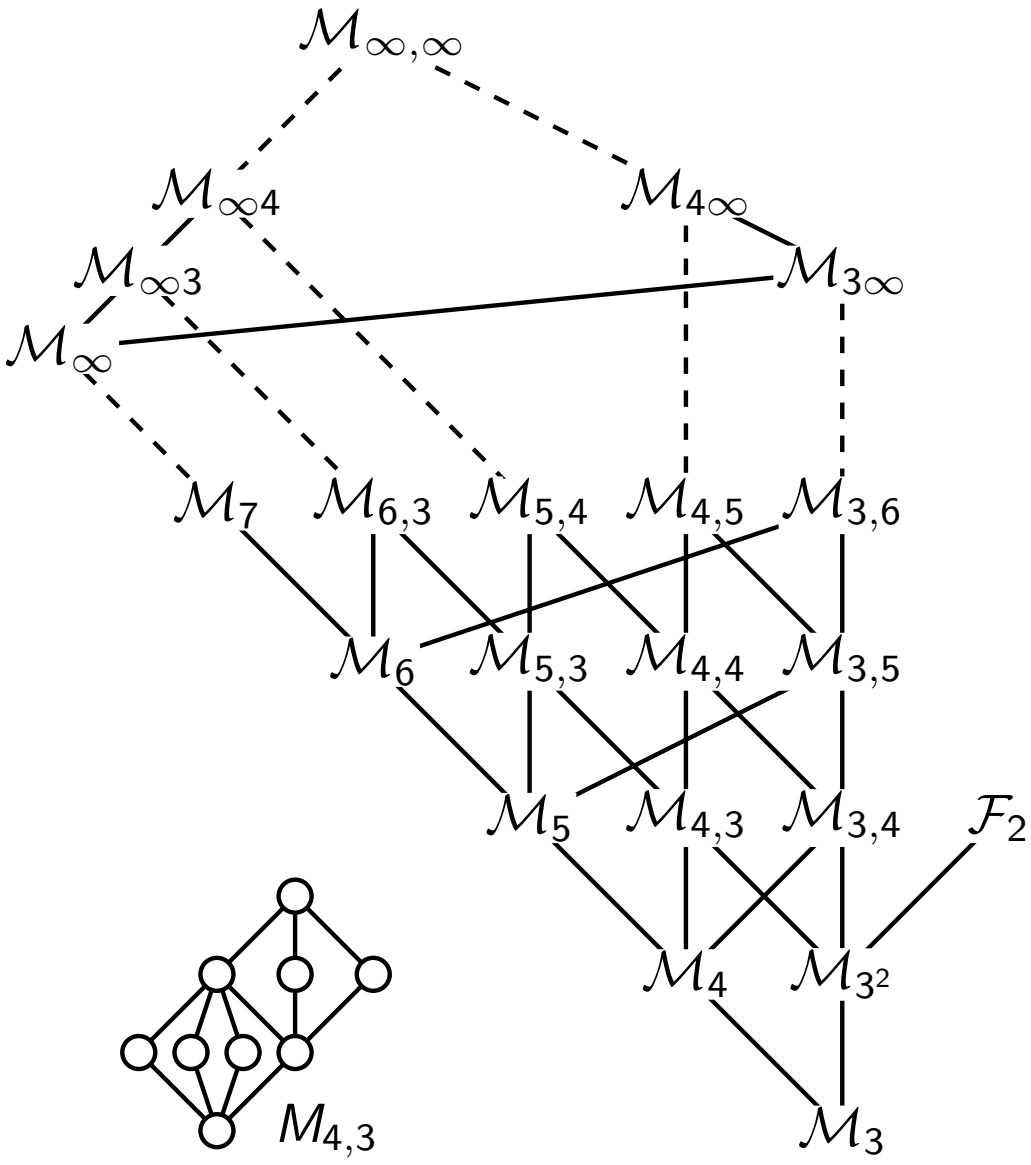
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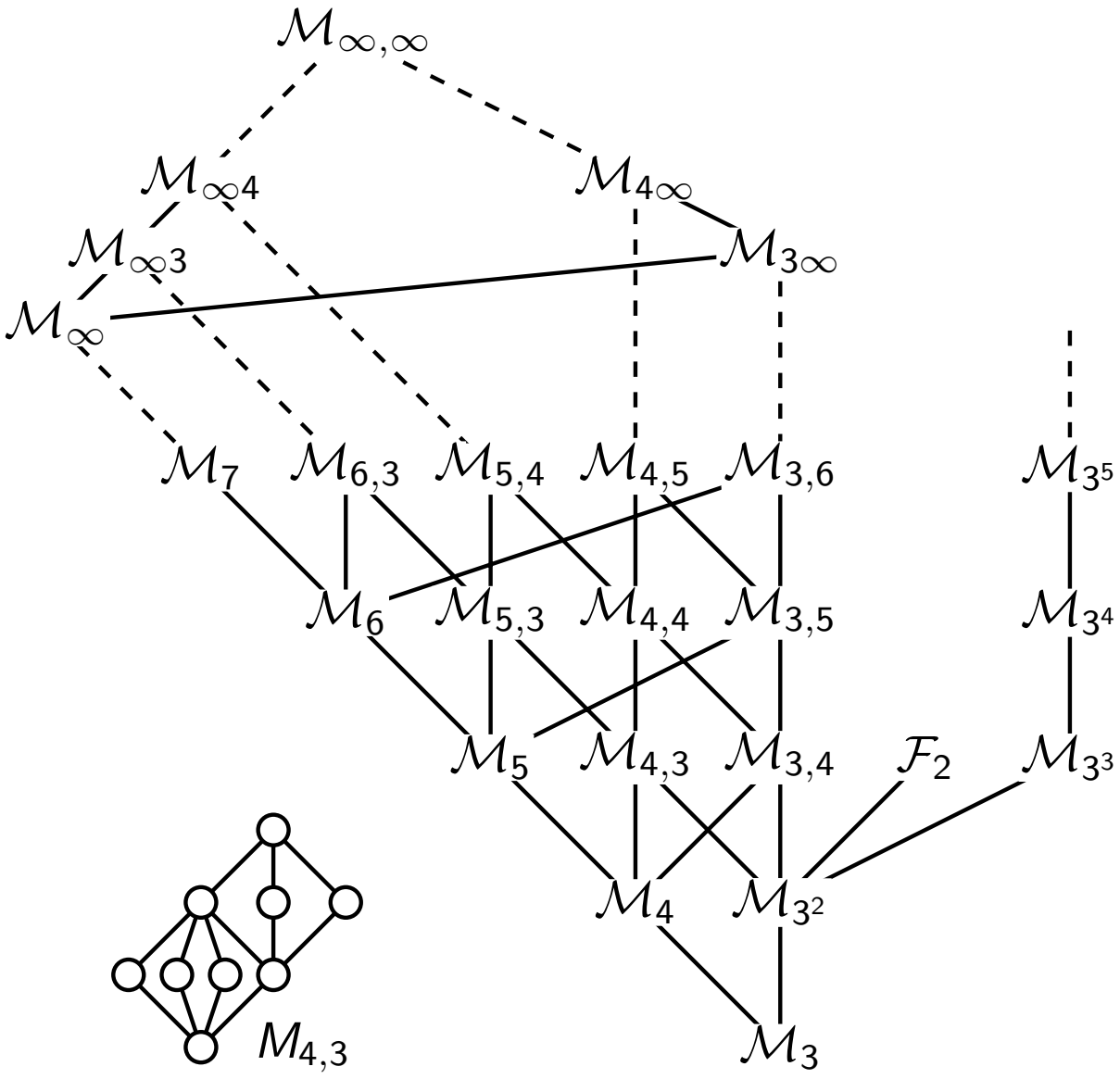
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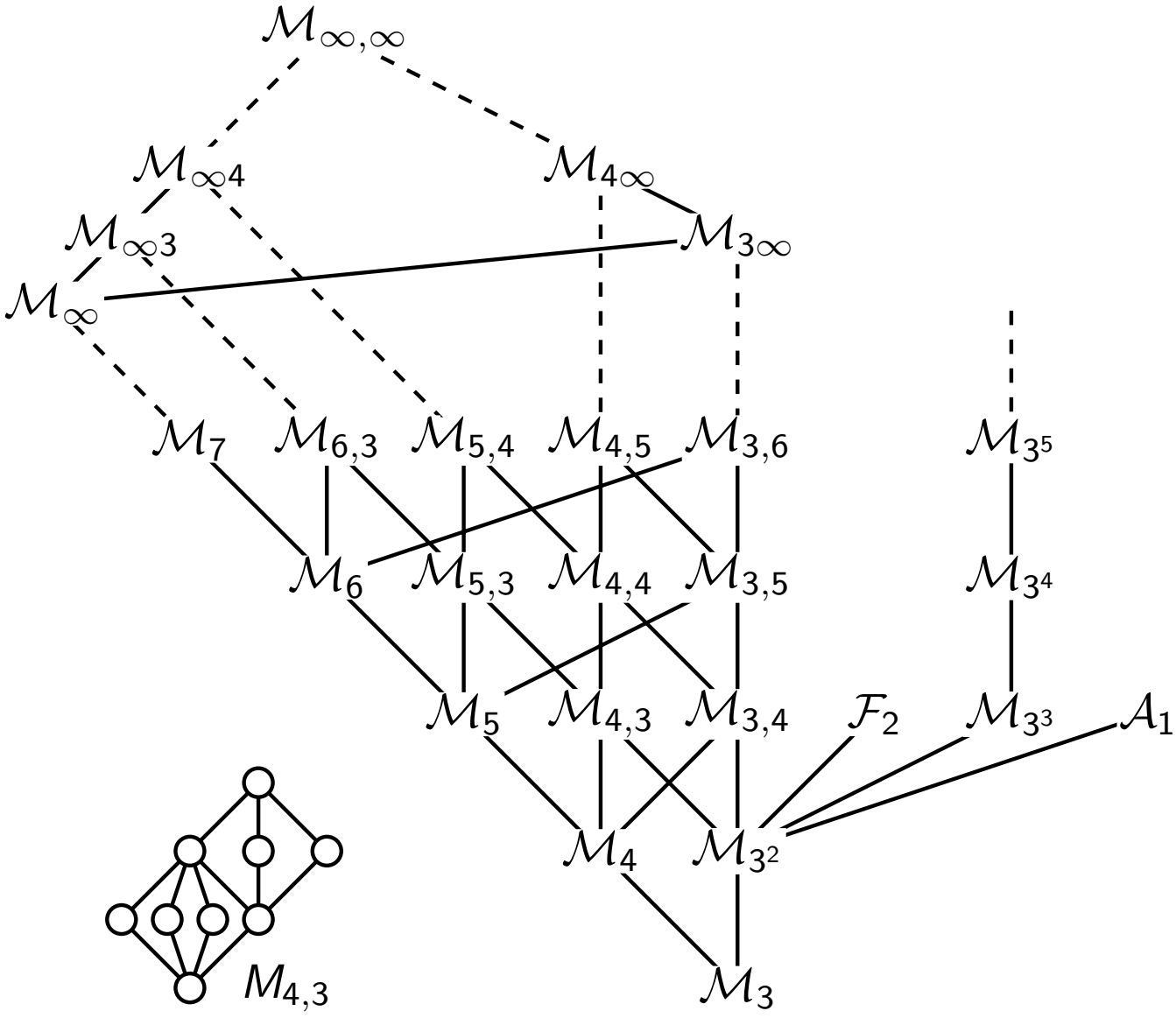
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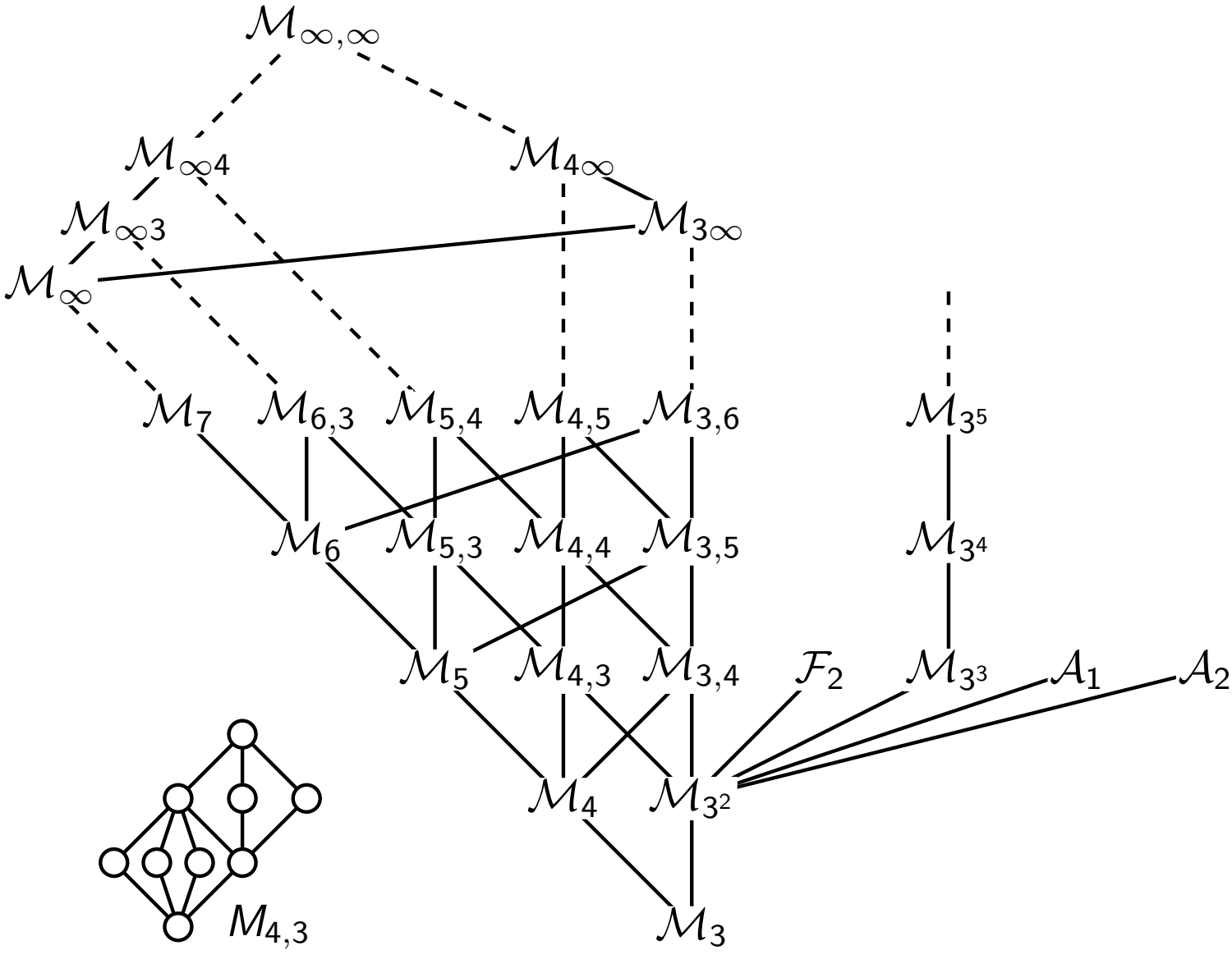
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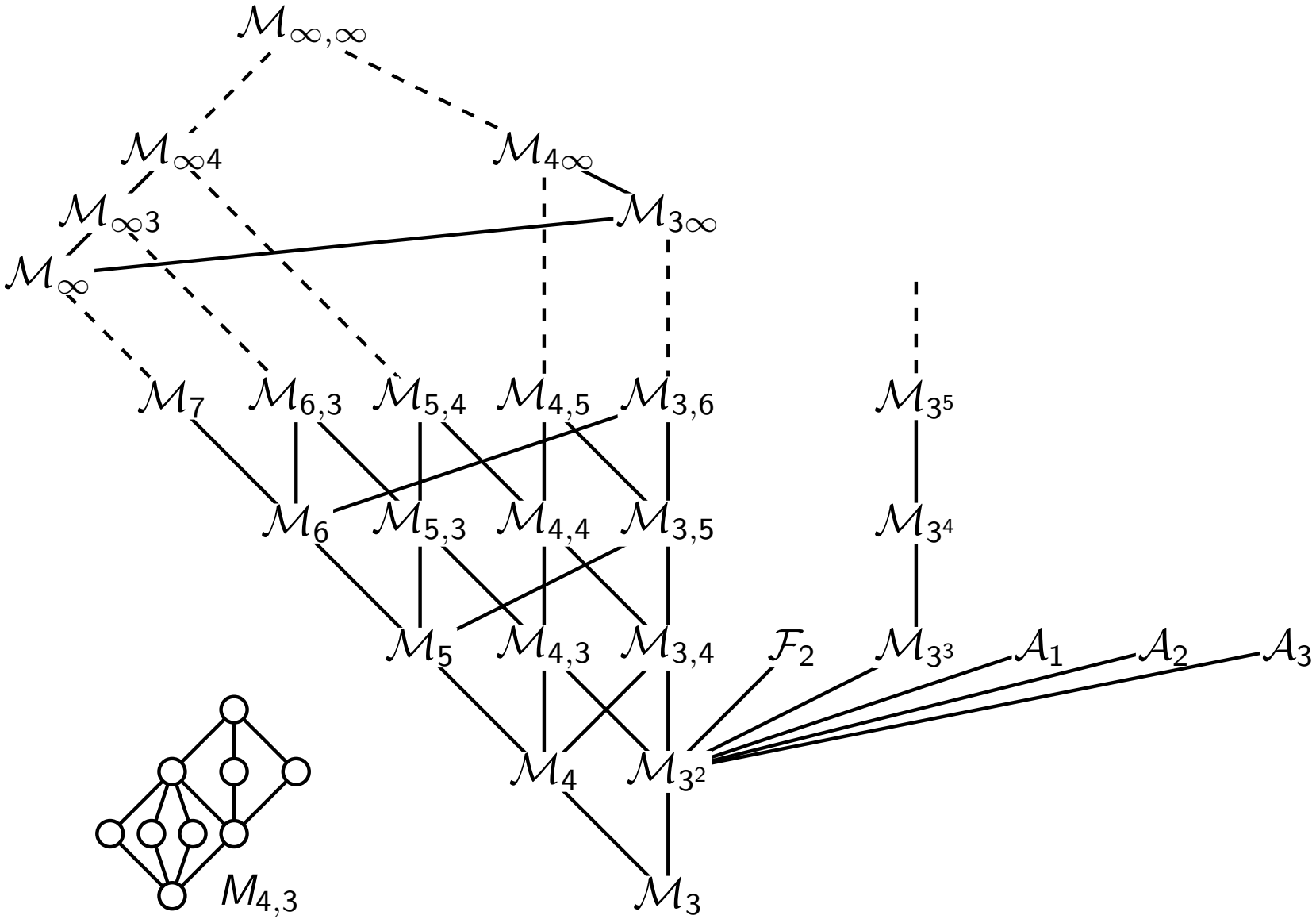
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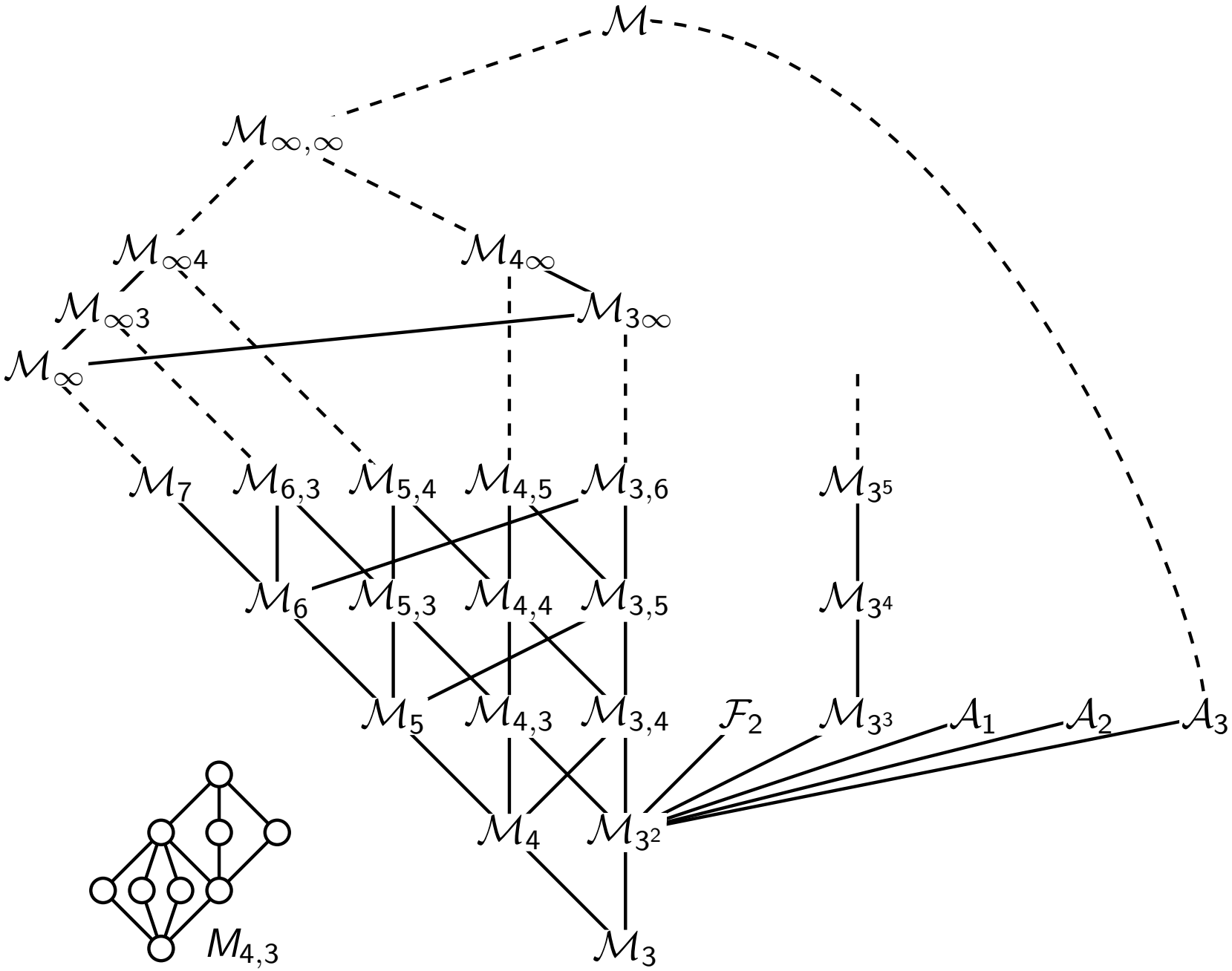
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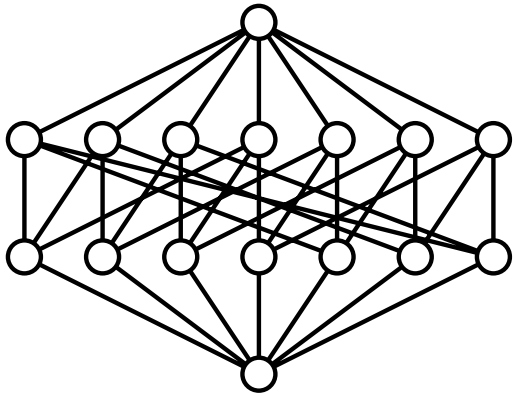
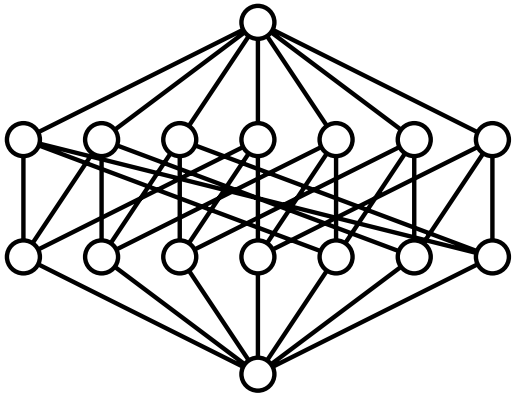


$F_2 = L(\mathbb{F}_2^3, \mathbb{F}_2)$



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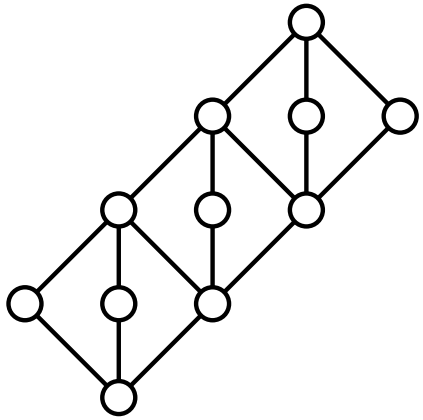
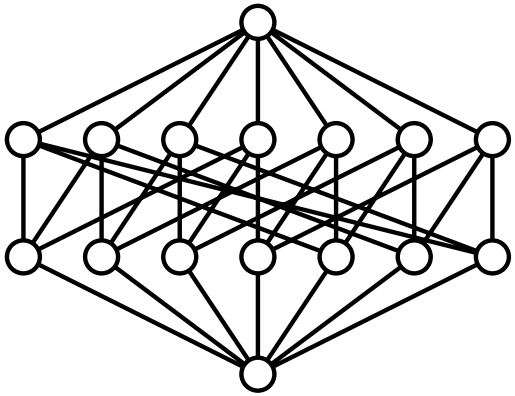


$M_{3^3}$



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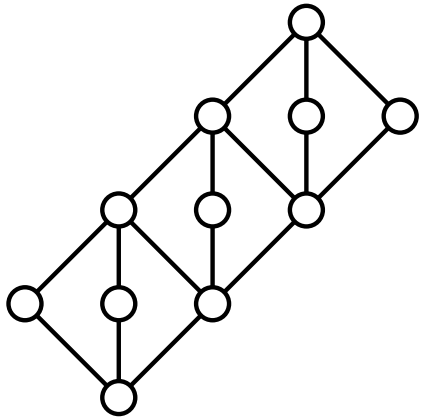

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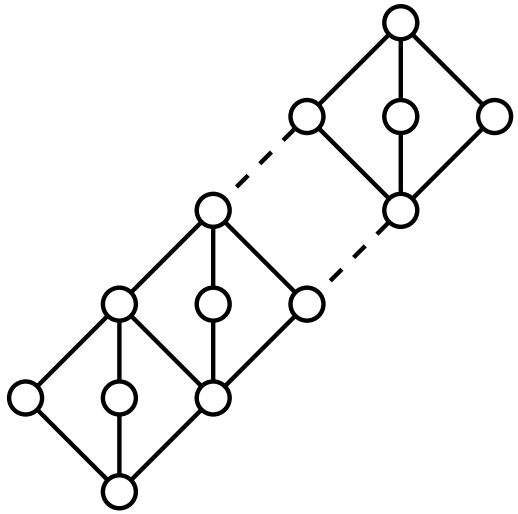

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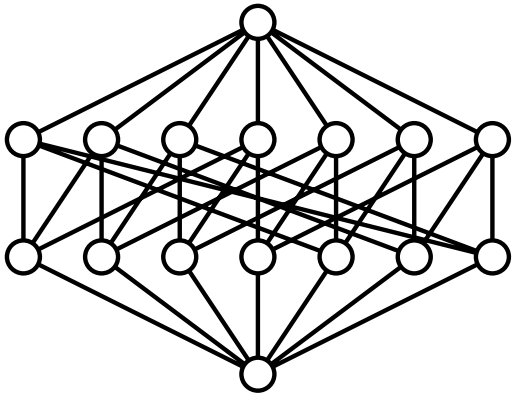
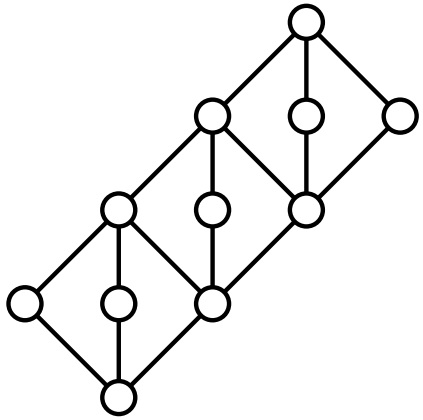


$F_2 = L(\mathbb{F}_2^3, \mathbb{F}_2)$



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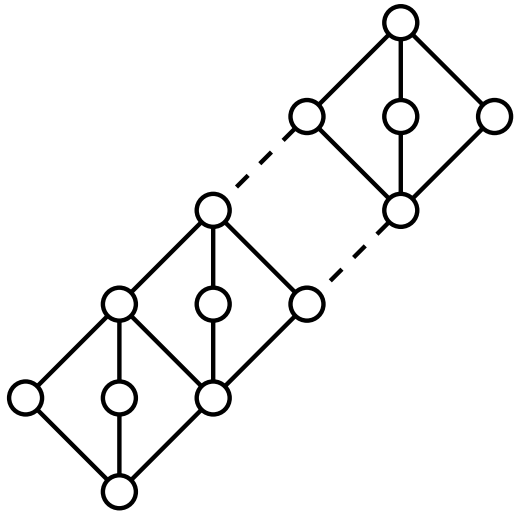

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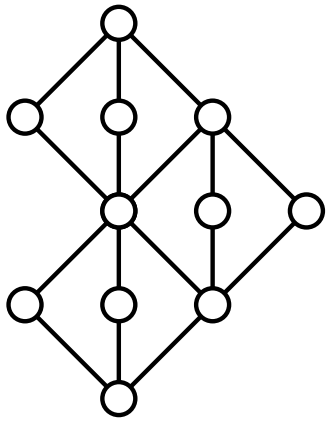

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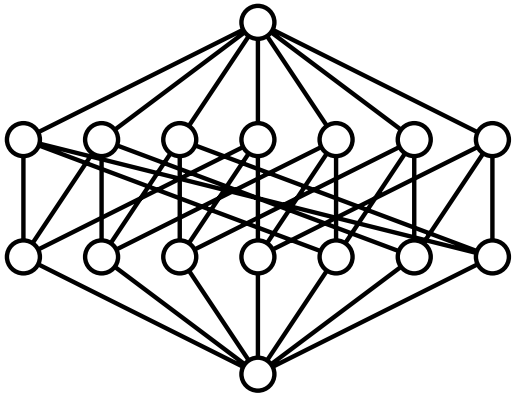
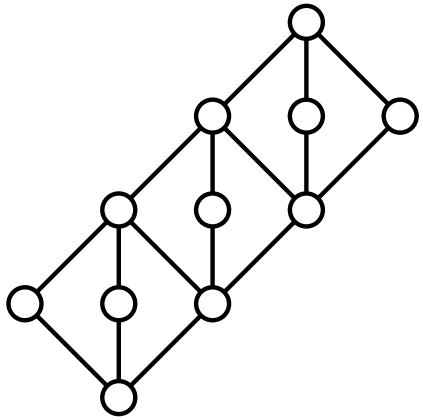


$F_2 = L(\mathbb{F}_2^3, \mathbb{F}_2)$



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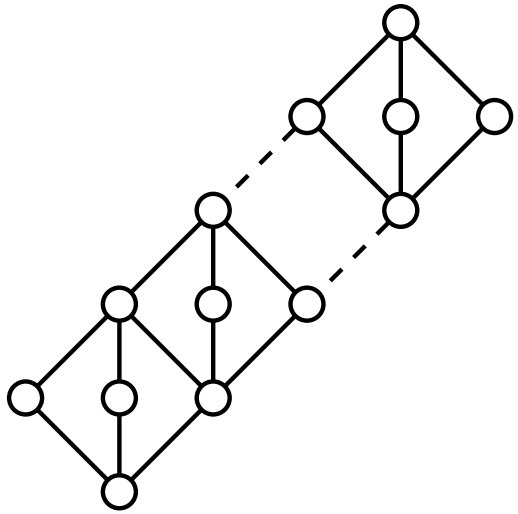
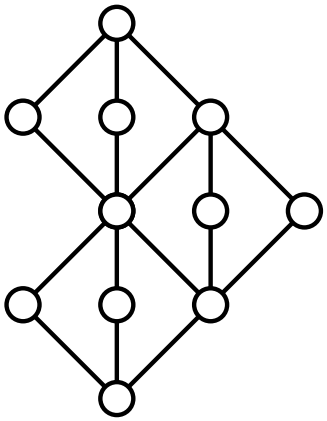
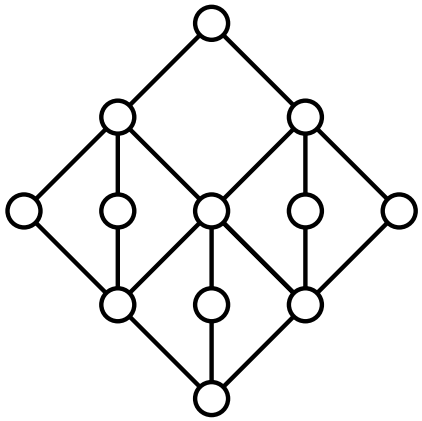


$M_{3^n}$



$A_1$



$A_2$

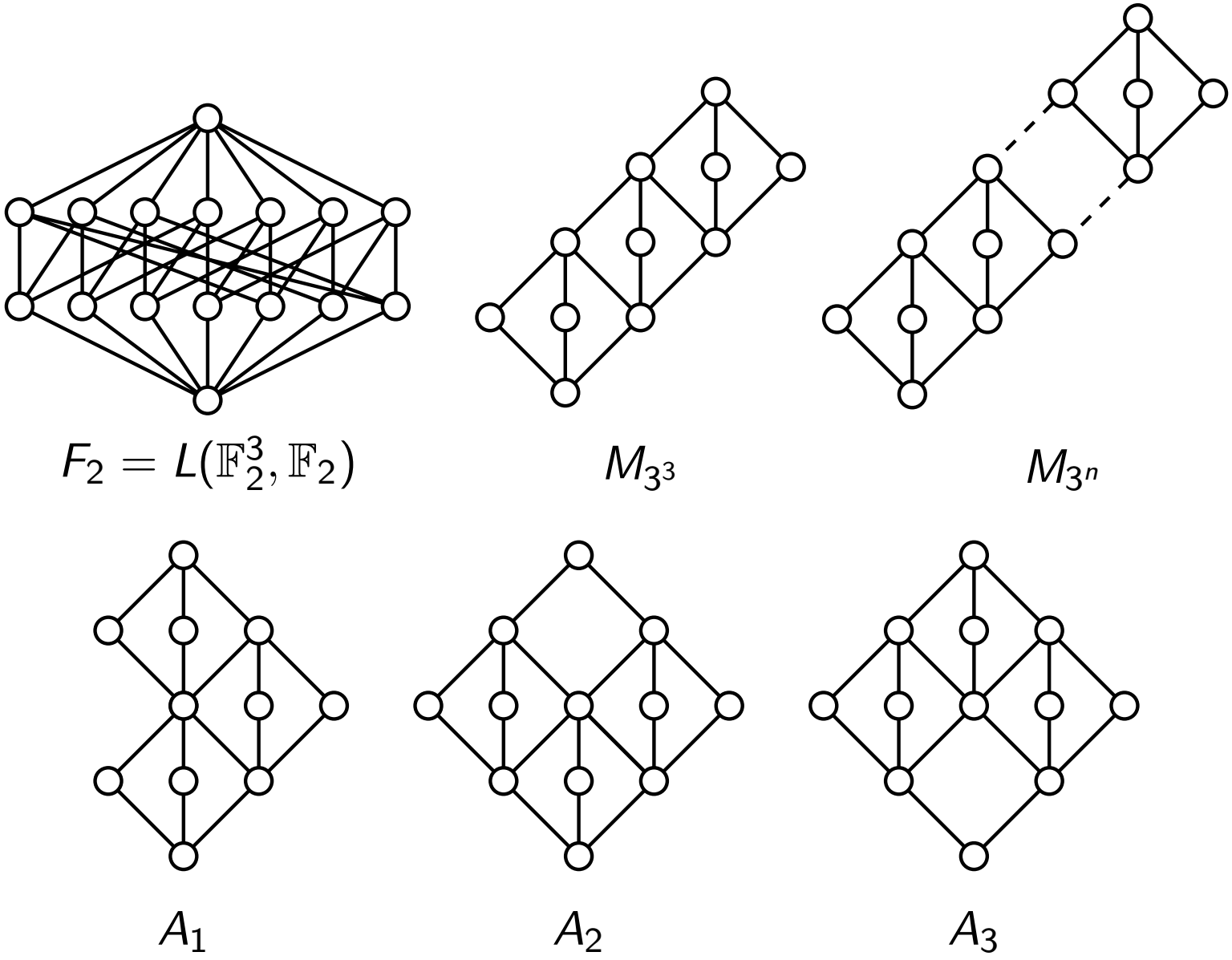



Figure: Modular lattices that generate varieties of finite height

# Varieties above $\mathcal{N}_5$

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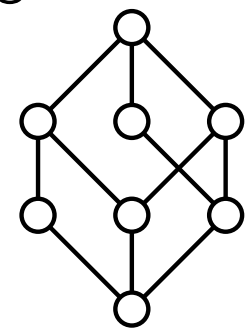
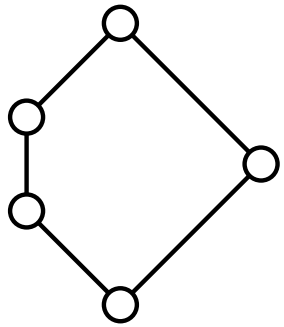
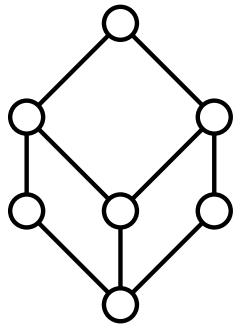
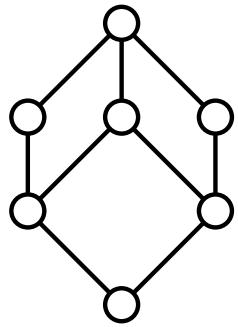
E.g. there is a join-irreducible variety above $L_1, L_2, L_4, L_5$ that is generated by 



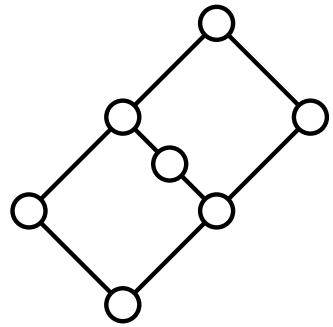
$N_5$



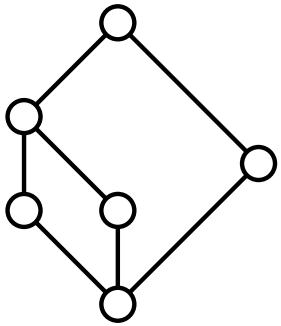
$L_1$



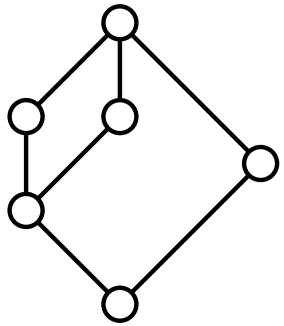
$L_2$



$L_3$



$L_4$



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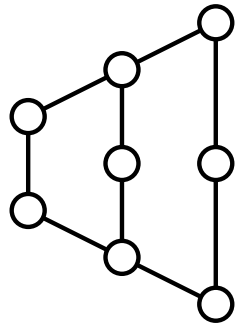
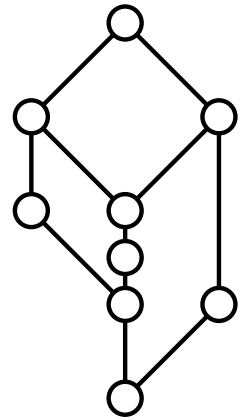
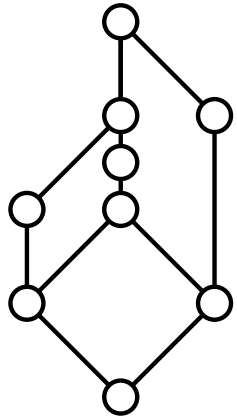


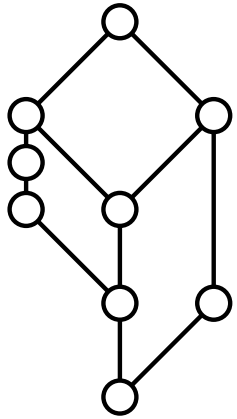
$L_6$



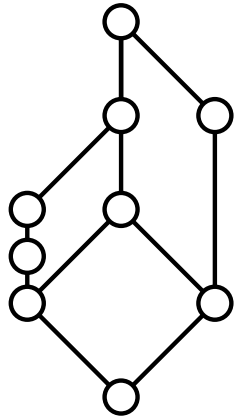
$L_7$



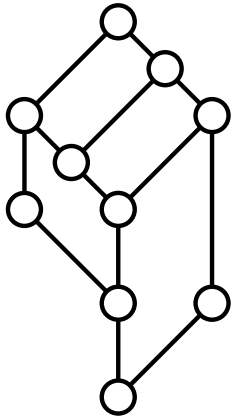
$L_8$



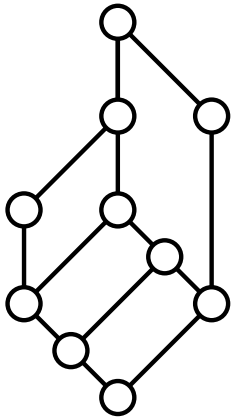
$L_9$



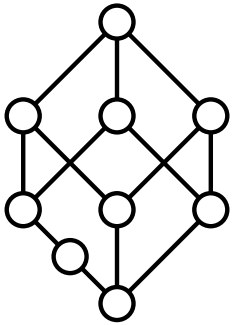
$L_{10}$



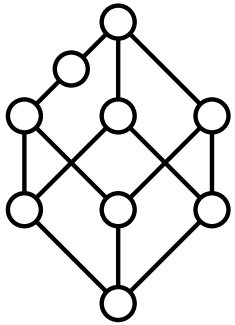
$L_{11}$



$L_{12}$



$L_{13}$



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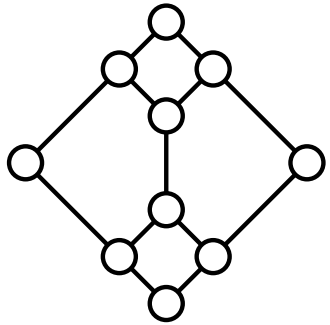

$L_{15}$

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Python is used to interact with and program in Sage

But Sage does not have direct support for Universal Algebra

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Has a Java graphical interface for displaying (congruence) lattices

But algebras have to be entered one at a time

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Sage can handle lists of algebras, e.g. compute Con($A$) for each of them

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The bottom lattice of lattice varieties is obtained from the downsets of this poset

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A *solution* of the CSP $(X, D, \mathcal{C})$ is a function $h : X \to D$ such that $\bar{h}(v_i) \in R_i$, where $\bar{h}(x_0, \ldots, x_{k-1}) = (h(x_0), \ldots, h(x_{k-1}))$

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|---|---|---|
| 1 | 1 | 0 |
| 2 | 1 | 1 |
| 3 | 1 | 0 |
| 4 | 2 | 0 |
| 5 | 5 | 2 |
| 6 | 15 | 4 |
| 7 | 53 | 16 |
| 8 | 222 | 69 |
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The poset has height 7 (so very flat and wide)

There are 2555 subdirectly irreducible lattices of size $\leq 10$

Each lattice generates a join-irreducible variety

Computing the poset of subvarieties took about 2 weeks

The poset has height 7 (so very flat and wide)

Levels 0, 1, 2 were discussed before (known) and confirmed by the calculations

# Future Work

Find a good way to view the data from these calculations

Apply some data mining techniques

Develop calculations that can prove the list of varieties above a given variety is complete

Find sets of equations that determine these varieties

# References

R. Freese, E. Kiss, M. Valeriote, Universal Algebra Calculator, http://uacalc.org

P. Jipsen and H. Rose, *Varieties of Lattices*, Lecture Notes in Mathematics, Springer Verlag, 1992, pp x+162

Minion Constraint Satisfier, http://minion.sourceforge.net

W. Stein, Sage Computer Algebra System, http://sagemath.org

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R. Freese, E. Kiss, M. Valeriote, Universal Algebra Calculator, http://uacalc.org

P. Jipsen and H. Rose, *Varieties of Lattices*, Lecture Notes in Mathematics, Springer Verlag, 1992, pp x+162

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Thank You