Proving the chain rule

Given $f'\big(g(x)\big)$ and $g'(x)$ exist, we want to find $\frac{d}{dx}\Big(f\big(g(x)\big)\Big)$.

Let $m(k) = \frac{f(g(x)+k)-f\left(g(x)\right)}{k}$ for $k \neq 0$ and $m(0) = f'\big(g(x)\big)$.

Then $\lim\limits_{k\to 0} m(k) = f'\big(g(x)\big)$, so m is **continuous** at 0.

Note that $f(g(x)+k) - f\big(g(x)\big) = m(k)k$ holds for **all** k.

Now let $k = g(x+h) - g(x)$, then $g(x) + k = g(x+h)$.

Hence $\frac{f\left(g(x+h)\right)-f\left(g(x)\right)}{h} = m\big(g(x+h) - g(x)\big)\frac{\left(g(x+h)-g(x)\right)}{h}$.

Taking limits as $h \to 0$ we get $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$.