# Residuated Lattices

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## Outline

Part I

- Universal Algebra
- Examples of residuated lattices
- Congruences and normal filters
- The lattice of subvarieties
- Varieties generated by positive universal classes
- Direct decompositions and poset products

Part II

- Residuated Frames
- Decidability
- Enumerating finite residuated lattices



## Set notation

We assume a background of basic *set notation* and *logic*

$A, B, C, \ldots$ denote *sets*; $x \in A$ means $x$ is an *element* of $A$

$\emptyset = \{\}$ is the *empty set*; *set-builder* notation: $\{x \mid P(x)\}$

*Intersection* $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

*Union* $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$\mathbb{N}$ = Natural numbers
$\mathbb{Z}$ = Integers
$\mathbb{R}$ = Real numbers

*Set difference* $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

*Ordered pairs* $(x, y) = (u, v) \iff x = u \text{ and } y = v$

*Cartesian product* $A \times B = \{(x, y) \mid x \in A \text{ and } x \in B\}$

Set of *n-tuples* $A^n = A \times A \times \cdots \times A$ ($n$ copies)



## Algebras and subalgebras

An $n$-ary *operation* on a set $A$ is a function $f : A^n \to A$

0-ary operations are *constants* (fixed elements of $A$)

An *algebra* $\mathbf{A} = (A, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \ldots)$ is a set $A$ with operations $f_i^{\mathbf{A}}$ of arity $n_i$

Superscript $^{\mathbf{A}}$ is useful when there are several algebras, otherwise omitted

The *type* of an algebra is the list of arities $(n_1, n_2, \ldots)$

E.g. a *group* $\mathbf{G} = (G, \cdot, ^{-1}, 1)$ is an algebra of type $(2, 1, 0)$

*Subsets:* $B \subseteq A$ means for all $x$, if $x \in B$ then $x \in A$

$g = f|_B$ means for all $b_i \in B$, $g(b_1, \ldots, b_n) = f(b_1, \ldots, b_n) \in B$

$\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B$ (all $i$)

## Homomorphisms and isomorphisms

Let $\mathbf{A}$, $\mathbf{B}$ be algebras of the same type

A *homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a function $h : A \to B$ such that for all $i$

$$h(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i}) = f_i^{\mathbf{B}}(h(a_1), \ldots, h(a_{n_i}))$$

$h$ is *onto* if $h[A] = \{h(a) \mid a \in A\} = B$

In this case $\mathbf{B} = h[\mathbf{A}]$ is called a *homomorphic image* of $A$

$h$ is *one-to-one* if for all $x, y \in A$, $x \neq y$ implies $h(x) \neq h(y)$

$h$ is an *isomorphism* if $h$ is a one-to-one and onto homomorphism

In this case $\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$



## Products and HSP

The *union* of sets $A_j$ $(j \in J)$ is $\bigcup_{j\in J} A_j = \{x \mid x \in A_j \text{ for some } j \in J\}$

$f : J \to \bigcup_{j\in J} A_j$ is a *choice function* if $f(j) \in A_j$ for all $j \in J$

The *cartesian product* $\prod_{j\in J} A_j$ is the set of all choice functions

The *direct product* of algebras $\mathbf{A}_j$ $(j \in J)$ is $\mathbf{A} = \prod_{j\in J} \mathbf{A}_j$ where
$A = \prod_{j\in J} A_j$ and $f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})(j) = f_i^{\mathbf{A}_j}(a_1(j), \ldots, a_{n_i}(j))$ for all $j \in J$

Let $\mathcal{K}$ be a class of algebras of the same type

$H\mathcal{K}$ is the class of homomorphic images of members of $\mathcal{K}$

$S\mathcal{K}$ is the class of algebras isomorphic to subalgebras of members of $\mathcal{K}$

$P\mathcal{K}$ is the class of algebras isomorphic to direct products of members of $\mathcal{K}$

$\mathcal{K}$ is a *variety* if $H(\mathcal{K}) = S(\mathcal{K}) = P(\mathcal{K}) = \mathcal{K}$ $\quad ( \overset{\text{Tarski}}{\iff} HSP(\mathcal{K}) = \mathcal{K})$



## Term algebras and equational classes

For a fixed type, the *terms with variables from a set $X$* is the smallest set $T(X)$ such that $X \subseteq T(X)$ and

$$\text{if } t_1, \ldots, t_{n_i} \in T(X) \text{ then } \text{“}f_i(t_1, \ldots, t_{n_i})\text{”} \in T(X) \text{ for all } i$$

The *term-algebra over $X$* is $\mathbf{T}(X) = (T(X), f_1^{\mathbf{T}}, f_2^{\mathbf{T}}, \ldots)$ with

$$f_i^{\mathbf{T}}(t_1, \ldots, t_{n_i}) = \text{“}f_i(t_1, \ldots, t_{n_i})\text{”} \quad \text{for all } i \text{ and } t_1, \ldots, t_{n_i} \in T(X)$$

An *equation* is a pair of terms $(s, t)$ written “$s{=}t$”; often omit “ ”

An *assignment* into an algebra $\mathbf{A}$ is a homomorphism $h : \mathbf{T}(X) \to \mathbf{A}$

An algebra $\mathbf{A}$ *satisfies* $s{=}t$ if $h(s) = h(t)$ for all assignments into $\mathbf{A}$

For a set $E$ of equations, $\mathrm{Mod}(E) = \{\mathbf{A} \mid \mathbf{A} \text{ satisfies } s{=}t \text{ for all } s{=}t \in E\}$

An *equational class* is of the form $\mathrm{Mod}(E)$ for some set of equations $E$



## Varieties and equational logic

Exercise: Show that every equational class is a variety

**Theorem (Birkhoff 1935)**

*Every variety is an equational class*

For a class $\mathcal{K}$ of algebras $\mathrm{Eq}(\mathcal{K}) = \{s{=}t \mid \mathbf{A} \text{ satisfies } s{=}t \text{ for all } \mathbf{A} \in \mathcal{K}\}$

An *equational theory* is of the form $\mathrm{Eq}(\mathcal{K})$ for some class of algebras $\mathcal{K}$

$t[x{\mapsto}r]$ is the term $t$ with all occurrences of $x$ replaced by the term $r$

**Theorem (Birkhoff 1935)**

*$E$ is an equational theory if and only if for all terms $q, r, s, t$*
$t{=}t \in E$; $\quad s{=}t \in E \implies t{=}s \in E$; $\quad r{=}s,\ s{=}t \in E \implies r{=}t \in E$
*and* $\quad q{=}r,\ s{=}t \in E \implies s[x{\mapsto}q]{=}t[x{\mapsto}r] \in E$

I.e. the rule of algebra: “replacing all $x$ by equals in equals gives equals”

## Examples of equational theories and varieties

A *binar* is an algebra $(A, \cdot)$ with one binary operation $x \cdot y$, written $xy$

A *semigroup* is an *associative* binar, i.e. satisfies $(xy)z = x(yz)$

A *band* is an *idempotent* semigroup, i.e. satisfies $xx = x$

A *semilattice* is a *commutative* band, i.e. satisfies $xy = yx$

A *unital binar* is an algebra $(A, \cdot, 1)$ that satisfies $1x = x$ and $x1 = x$

A *monoid* is a unital binar that is associative, i.e. a unital semigroup

$(A, \cdot, ^{-1}, 1)$ is a *group* if $\cdot$ is associative, $1x = x$ and $x^{-1} \cdot x = 1$

Exercise: Show that a group satisfies $x1 = x$, $xx^{-1} = 1$ and $(x^{-1})^{-1} = x$

Hint: $x = 1x = (x^{-1})^{-1}x^{-1}x = (x^{-1})^{-1}1 = (x^{-1})^{-1}11 = (x^{-1})^{-1}x^{-1}x1 = 1x1 = x1$



## Binary relations and partial orders

$R$ is a *binary relation on a set* $A$ if it is a subset of $A \times A$

E.g. the *identity relation* $id_A = \{(a, a) \mid a \in A\}$ is a binary relation on $A$

$$aRb \quad \text{means} \quad (a, b) \in R$$

$R$ is *reflexive* if $xRx$ for all $x \in A$

$R$ is *antisymmetric* if $xRy$ and $yRx$ implies $x = y$

$R$ is *transitive* if $xRy$ and $yRz$ implies $xRz$

$R$ is a *partial order* if it is reflexive, antisymmetric and transitive

For a semilattice $\mathbf{A}$ define $a \leq^{\mathbf{A}} b \iff ab = a$

Exercise: Prove that $\leq^{\mathbf{A}}$ is a partial order on $A$



## Posets and meet-semilattices

A *poset* $(A, \leq)$ is a set $A$ with a partial order $\leq$ on $A$

For $S \subseteq A$ the *meet* $\bigwedge S$ is defined by $x \leq \bigwedge S \iff x \leq s$ for all $s \in S$

Exercise: Prove $\bigwedge S$ is unique (if it exists; $\bigwedge$ = greatest lower bound)

$\bigwedge\{x, y\}$ is denoted by $x \wedge y$

Exercise: Prove that in a semilattice $ab = a \wedge b$ for the partial order $\leq^{\mathbf{A}}$

A *meet-semilattice* is a poset in which $a \wedge b$ exists for all $a, b$

A meet-semilattice is *complete* if $\bigwedge S$ exists for all nonempty subsets $S$

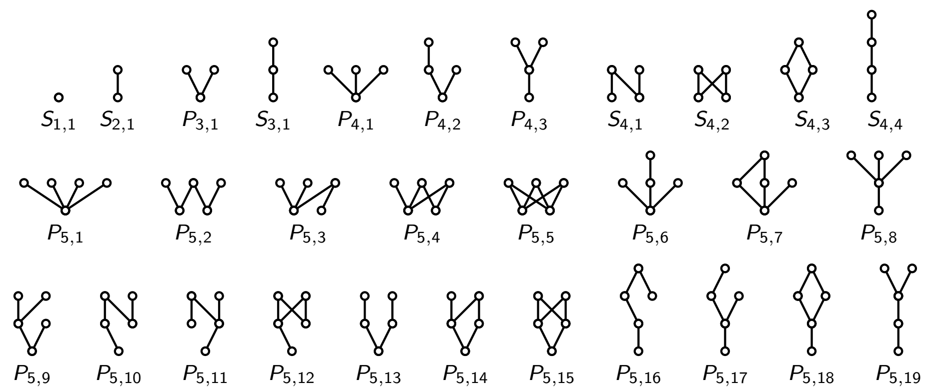
Exercise: Prove if $(A, \leq)$ is a meet-semilattice then $(A, \wedge)$ is a semilattice

An element $a$ has a *cover* $b$, denoted $a \prec b$, if $\{x \mid a \leq x \leq b\} = \{a, b\}$

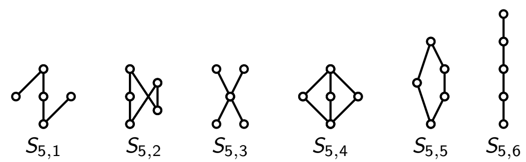
A *Hasse diagram* of a poset has an upward line from dot $a$ to $b$ if $a \prec b$



## (Dually)nonisomorphic connected posets with $\leq 5$ elements




$P_{n,k}$ = nonselfdual
$S_{n,k}$ = selfdual
$n$-element poset




Exercise: Which posets are meet-semilattices?

## Lattices

For a relation $\leq$, define the *dual* $\geq$ by $\quad b \geq a \iff a \leq b$

$\mathbf{A}^\partial = (A, \geq)$ is the *dual poset* of $\mathbf{A} = (A, \leq)$

Every partial order concept has a dual, obtained by interchanging $\leq$ and $\geq$

The *join* $\bigvee$ is defined dually to the meet $\bigwedge$ (join = least upper bound)

A *join-semilattice* is a poset where $a \vee b = \bigvee\{a, b\}$ exists for all $a, b$

A *lattice* is a poset that is a meet-semilattice and a join-semilattice

Note $x \leq y$ is definable by $x \vee y = y$, as well as by $x \wedge y = x$

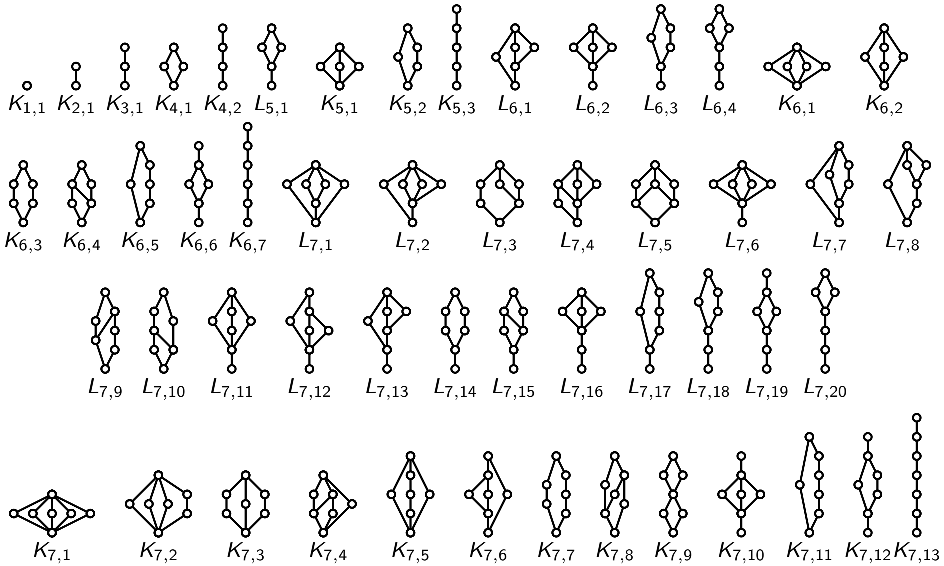
Exercise: Show $\mathbf{A} = (A, \wedge, \vee)$ is a lattice iff $\wedge, \vee$ are associative, commutative and *absorbtive*, i.e. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

Hint: $x \wedge x = x \wedge (x \vee (x \wedge y)) = x$



## All (dually-)nonisomorphic lattices with $\leq 7$ elements

$K_{n,k}$ = selfdual lattice of size $n$ $\qquad$ $L_{n,k}$ = nonselfdual lattices of size $n$






## Examples of lattices

A lattice is *complete* if $\bigwedge S$ and $\bigvee S$ exist for all subsets $S$

A lattice is *bounded* if it has a top element $\top$ and a bottom element $\bot$

Exercise: Show that every complete lattice is bounded

Exercise: Show any complete meet-semilattice with $\top$ is a complete lattice

The *powerset* $\mathcal{P}(X)$ of all subsets of $X$ is a complete lattice with $\bigcap, \bigcup$

The collection $\Lambda_\mathcal{V}$ of *subvarieties* of $\mathcal{V}$ is a complete lattice with $\bigwedge = \bigcap$

Any *linear order* (i.e. $x \leq y$ or $y \leq x$ for all $x, y$) is a lattice

A lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds ($\Leftrightarrow$ dual)

E.g. $\mathcal{P}(X)$ and any linear order are distributive lattices



## Equivalence relations and congruences

Let $\mathbf{A}$ be an algebra and $R$ a binary relation on $A$

$R$ is *symmetric* if $xRy$ implies $yRx$ (implicitly quantified)

$R$ is an *equivalence relation* if it is reflexive, symmetric and transitive

$R$ is a *congruence* on $\mathbf{A}$ if it is an equivalence relation and

$xRy$ implies $f_i(a_1, \ldots, x, \ldots, a_n)\, R\, f_i(a_1, \ldots, y, \ldots, a_n)$ (all args, $i$)

The set $\mathrm{Con}(\mathbf{A})$ of *all congruences* on $\mathbf{A}$ is a complete lattice with $\bigwedge = \bigcap$

$\bot = id_A$ and $\top = A^2$; $\quad \mathrm{con}(a, b) = \bigcap\{R \in \mathrm{Con}(\mathbf{A}) \mid aRb\}$

A *congruence class* is a set of the form $[a]_R = \{x \mid aRx\}$

$\{C_i : i \in I\}$ is a *partition* of $A$ if $A = \bigcup_{i \in I} C_i$ and $C_i \cap C_j = \emptyset$ or $C_i = C_j$

The set $A/R$ of *all congruence classes* is a partition of $A$

## Homomorphic images and quotient algebras

The *quotient algebra* $\mathbf{A}/R = (A/R, f_1, f_2, \dots)$ is defined by

$$f_i([a_1]_R, \dots, [a_{n_i}]_R) = [f_i^{\mathbf{A}}(a_1, \dots, a_{n_i})]_R$$

Exercise: Show that $f_i$ is well-defined if and only if $R$ is a congruence

For a homom. $h : A \to B$, define the *kernel* $\ker h = \{(x, y) \mid h(x) = h(y)\}$

Exercise: Show that $\ker h$ is a congruence on $\mathbf{A}$ and that

the *natural map* $[\_]_R : \mathbf{A} \to \mathbf{A}/R$ is a homomorphism

**Theorem (First Isomorphism Theorem)**

*$k : \mathbf{A}/\ker h \to h[\mathbf{A}]$ defined by $k([a]_{\ker h}) = h(a)$ is an isomorphism*

**Theorem (Second Isomorphism Theorem)**

*If $R \subseteq S$ are congruences on $\mathbf{A}$ and $T = \{([a]_R, [b]_R) \mid aSb\}$ then $T \in Con(\mathbf{A}/R)$ and $(\mathbf{A}/R)/T \cong \mathbf{A}/S$*



## Subdirectly irreducible algebras

An algebra is *directly decomposable* if it is isomorphic to a direct product of nontrivial algebras (happens rarely)

Let $R_j \in \mathrm{Con}(\mathbf{A})$ and define $h : \mathbf{A} \to \prod_{j \in J} \mathbf{A}/R_j$ by $h(a)(j) = [a]_{R_j}$

Exercise: Show that $h$ is one-to-one if and only if $\bigcap_{j \in J} R_j = id_A$

In this case $h$ is called a *subdirect decomposition* of $\mathbf{A}$

An element $c$ in a complete lattice is *completely meet irreducible* if $c = \bigwedge S$ implies $c \in S$ for all subsets $S$; equiv. if $c$ has a unique cover

$\mathbf{A}$ is *subdirectly irreducible* if $id_A$ is completely meet irreducible in $\mathrm{Con}(\mathbf{A})$

**Theorem (Birkhoff 1944)**

*Every algebra $\mathbf{A}$ has a subdirect decomposition using only subdirectly irreducible homomorphic images of $\mathbf{A}$*



Birkhoff's Theorem says that every algebra is a subalgebra of a product of subdirectly irreducible algebras (s.i. algebras for short)

So the s.i. algebras are building blocks of varieties

For an element $a$ in a poset, the *principal filter of* $a$ is $\uparrow a = \{x \mid a \le x\}$

A subset $S$ of a poset is an *upset* if for all $a \in S$ we have $\uparrow a \subseteq S$

$S$ is *down-directed* if for all $a, b \in S$ there is a $c \in S$ with $c \le a$ and $c \le b$

A *filter* $F$ is a down-directed upset

An *ideal* is the dual concept of a filter, i.e. an *up-directed downset*

Exercise: Show that the 2-element semilattice is the only s.i. semilattice and that the 2-element lattice is the only s.i. distributive lattice

Hint: Every congruence is the intersection of congruences with two blocks

Every semilattice is a subalgebra of a product of two-element semilattices



## Introductory references for further background reading

Garrett Birkhoff, "**Lattice Theory**", 3rd ed., AMS Colloquium Publications, Vol. 25, 1967

Stan Burris and H. P. Sankapannavar, "**A Course in Universal Algebra**", Springer-Verlag, 1981, online at `www.math.uwaterloo.ca/~snburris/`

Brian Davey and Hilary Priestley, "**Introduction to Lattices and Order**", 2nd ed, Cambridge University Press, 2002

Nick Galatos, Peter Jipsen, Tomasz Kowalski and Hiroakira Ono, "**Residuated Lattices: an algebraic glimpse at substructural logics**", Studies in Logics and the Foundations of Mathematics, Elsevier, 2007

## Residuated maps

A function $f : (A, \leq) \to (B, \leq)$ is *residuated* if some $g : B \to A$ satisfies

$$f(x) \leq y \iff x \leq g(y) \qquad \text{for all } x \in A,\ y \in B$$

$g$ is called the *residual of f* and $g(y) = \max\{x \mid f(x) \leq y\}$ (if it exists)

Exercise: $f$ preserves all existing joins and $g$ preserves all existing meets

Exercise: Show that $f$ is residuated $\iff$ $f$, $g$ are order-preserving and

$$f(g(y)) \leq y \text{ and } x \leq g(f(x)) \qquad \text{for all } x \in A,\ y \in B$$

Exercise: If $\mathbf{A}$, $\mathbf{B}$ are lattices then $f$ is residuated with residual $g$ $\iff$

$$f(x \wedge g(y)) \vee y = y \text{ and } x = x \wedge g(f(x) \vee y) \text{ hold}$$

Hence residuation can be expressed by equations (also in semilattices)



## Heyting algebras and Boolean algebras

$\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ is a *Heyting algebra* (HA) if

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and
- for all $a, b, c \in A$, $a \wedge b \leq c \iff b \leq a \to c$ ($\wedge$-residuation)

The unary *negation* operation $\neg$ is defined by $\neg x = x \to 0$

A *Boolean algebra* (BA) is a Heyting algebra that satisfies $\neg\neg x = x$

Exercise: Heyting algebras are distributive lattices and $x \wedge \neg x = 0$
BAs are also defined by $x \vee \neg x = 1$, i.e. $\neg$ is a *complement*
$\wedge$-residuation can be written equationally (hint: $a \wedge \_$ has residual $a \to \_$)

BAs give algebraic semantics for classical propositional logic $x \to y = \neg x \vee y$

HAs give algebraic semantics for intuitionistic propositional logic



## Algebras of relations

For binary relations $R$ and $S$ on a set $X$, we denote by

- $R'$ the complement $X^2 - R$ and by $R^\smile$ the *converse* $\{(y, x) \mid xRy\}$
- $R \cdot S$ the *composition* $\{(x, y) \mid (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z\}$
- $R\backslash S = (R^\smile \cdot S')'$ and $S/R = (S' \cdot R^\smile)'$
- $R \to S = R' \cup S$

Exercise: Check that

- $(\mathcal{P}(X^2), \cap, \cup, \to, \emptyset, X^2)$ is a Boolean algebra
- $(\mathcal{P}(X^2), \cdot, id_X)$ is a monoid
- for all $R, S, T \subseteq X^2$,

$$R \cdot S \subseteq T \iff S \subseteq R\backslash T \iff R \subseteq T/S$$



## Relation algebras

A relation algebra is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, {}')$ with

- $x \to y = x' \vee y$, $\perp = 1 \wedge 1'$ and $\top = 1 \vee 1'$ such that
- $(A, \wedge, \vee, \to, \perp, \top)$ is a Boolean algebra
- $(A, \cdot, 1)$ is a monoid
- for all $a, b, c \in A$,

$$a \cdot b \leq c \iff b \leq a\backslash c \iff a \leq c/b \qquad \text{(residuation)}$$

- $x^{\smile\smile} = x$ and $(x \cdot y)^\smile = y^\smile \cdot x^\smile$ where $x^\smile$ is defined as $(x\backslash 1')'$

Exercise: Show that $x^\smile = (1'/x)'$ and $x^\smile(xy)' \leq y'$

## $\ell$-groups

A lattice-ordered group is a lattice with an order-pres. group operation

Alternatively, a *lattice-ordered group* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice
- $(L, \cdot, 1)$ is a monoid
- for all $a, b, c \in L$,

$$a \cdot b \leq c \iff b \leq a\backslash c \iff a \leq c/b$$

- and $\mathbf{L}$ satisfies $(1/x) \cdot x = 1$

Exercise: Show that $(L, \cdot, ^{-1}, 1)$ is a group, where $x^{-1} = 1/x = x\backslash 1$

Example. The reals $\mathbb{R}$ under the usual order, addition and subtraction



## Powerset of a monoid

Let $\mathbf{M} = (M, \cdot, e)$ be a monoid and $X, Y \subseteq M$ and define

$X \cdot Y = \{x \cdot y \mid x \in X \text{ and } y \in Y\}$
$X\backslash Y = \{z \in M \mid X \cdot \{z\} \subseteq Y\}$
$Y/X = \{z \in M \mid \{z\} \cdot X \subseteq Y\}$

Exercise: Show that the powerset $\mathcal{P}(M)$ satisfies

- $(\mathcal{P}(M), \cap, \cup)$ is a lattice
- $(\mathcal{P}(M), \cdot, \{e\})$ is a monoid
- for all $X, Y, Z \subseteq M$,

$$X \cdot Y \subseteq Z \iff Y \subseteq X\backslash Z \iff X \subseteq Z/Y$$



## Ideals of a ring

Let $\mathbf{R}$ be a ring with unit and let $\mathcal{I}(\mathbf{R})$ be the set of (two-sided) ideals of $\mathbf{R}$

For $I, J \in \mathcal{I}(\mathbf{R})$ define $I \cdot J = \{a_1b_1 + \cdots + a_nb_n \mid a_i \in I, b_i \in J\}$

$I\backslash J = \{a \in R \mid Ia \subseteq J\}$ $\qquad$ $J/I = \{a \in R \mid aI \subseteq J\}$

$I \vee J = \{a + b \mid a \in I \text{ and } b \in J\}$

Exercise: Show that the set of ideal $\mathcal{I}(\mathbf{R})$ is closed under $\backslash, /$ and

- $(\mathcal{I}(\mathbf{R}), \cap, \vee)$ is a lattice
- $(\mathcal{I}(\mathbf{R}), \cdot, R)$ is a monoid
- for all ideals $I, J, K$ of $\mathbf{R}$

$$I \cdot J \subseteq K \iff J \subseteq I\backslash K \iff I \subseteq K/J$$



## Residuated lattices

A *residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$, $\quad ab \leq c \iff b \leq a\backslash c \iff a \leq c/b$

A **F***ull* **L***ambek algebra* is a residuated lattice with a new constant $0$

In an FL-algebra, define two *linear negations* $\sim x = x\backslash 0$ and $-x = 0/x$

A FL-algebra or residuated lattice is called

- commutative if $(L, \cdot, 1)$ is commutative ($xy = yx$)
- distributive if $(L, \wedge, \vee)$ is distibutive
- integral if it satisfies $x \leq 1$
- contractive if it satisfies $x \leq x^2$
- involutive if it satisfies $\sim -x = x = -\sim x$

## Properties

1. $x(y \vee z) = xy \vee xz$ and $(y \vee z)x = yx \vee zx$
2. $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ and $(y \wedge z)/x = (y/x) \wedge (z/x)$
3. $x/(y \vee z) = (x/y) \wedge (x/z)$ and $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$
4. $(x/y)y \leq x$ and $y(y\backslash x) \leq x$
5. $x(y/z) \leq (xy)/z$ and $(z\backslash y)x \leq z\backslash(yx)$
6. $(x/y)/z = x/(zy)$ and $z\backslash(y\backslash x) = (yz)\backslash x$
7. $x\backslash(y/z) = (x\backslash y)/z$
8. $x/1 = x = 1\backslash x$
9. $1 \leq x/x$ and $1 \leq x\backslash x$
10. $x \leq y/(x\backslash y)$ and $x \leq (y/x)\backslash y$
11. $y/((y/x)\backslash y) = y/x$ and $(y/(x\backslash y))\backslash y = x\backslash y$
12. $x/(x\backslash x) = x$ and $(x/x)\backslash x = x$
13. $(z/y)(y/x) \leq z/x$ and $(x\backslash y)(y\backslash z) \leq x\backslash z$

Multiplication is order preserving in both arguments. Each division operation is order preserving in the numerator and order reversing in the denominator.



Proofs of some of these properties

$$\begin{aligned} x(y \vee z) \leq w &\Leftrightarrow y \vee z \leq x\backslash w \\ &\Leftrightarrow y \leq x\backslash w \text{ and } z \leq x\backslash w \\ &\Leftrightarrow xy \leq w \text{ and } xz \leq w \\ &\Leftrightarrow xy \vee xz \leq w \end{aligned}$$

$$x/y \leq x/y \Rightarrow (x/y)y \leq x$$

$$x(y/z)z \leq xy \Rightarrow x(y/z) \leq (xy)/z$$

$$[(x/y)/z](zy) \leq x \Rightarrow (x/y)/z \leq x/(zy)$$

$$[x/(zy)]zy \leq x \Rightarrow x/(zy) \leq (x/y)/z$$

$$\begin{aligned} w \leq x\backslash(y/z) &\Leftrightarrow xw \leq y/z \\ &\Leftrightarrow xwz \leq y \\ &\Leftrightarrow wz \leq x\backslash y \\ &\Leftrightarrow w \leq (x\backslash y)/z \end{aligned}$$



## Lattice/monoid properties

$$(z/y)(y/x)x \leq (z/y)y \leq z \Rightarrow (z/y)(y/x) \leq z/x$$

RL's satisfy no special purely lattice-theoretic or monoid-theoretic property

Every lattice can be embedded in a (cancellative) residuated lattice

Every monoid can be embedded in a (distributive) residuated lattice



## Congruences in groups and Boolean algebras

Recall that a *congruence* on an algebra **A** is an equivalence relation on $A$ that is compatible with the operations of **A**

A subgroup $N$ of a group **G** is *normal* if $a \in N$ implies $x^{-1}ax \in N$

Congruences in groups correspond to normal subgroups:

For a congruence $R$ on **G**, the congruence class $[1]_R$ is a normal subgroup

Given a normal subgroup $N$ of a group **G**, the relation $R_N$ is a congruence, where $(a, b) \in R_N$ iff $a\backslash b \in N$ iff $\{a\backslash b, b\backslash a\} \subseteq N$

Congruences in Boolean algebras correspond to filters:

Given a congruence $R$ on a BA, the congruence class $[1]_R$ is a filter of **A**

Given a filter $F$ of a Boolean algebra **A**, $R_F$ is a congruence, where $(a, b) \in R_F$ iff $a \leftrightarrow b \in F$ iff $\{a \to b, b \to a\} \subseteq F$

## Congruences on rings and monoids

Congruences on rings correspond to (two-sided) ideals

Congruences on $\ell$-groups correspond to convex $\ell$-subgroups

Congruences on monoids do not correspond to any particular kind of subset

Do congruences on residuated lattices correspond to certain subsets?



## Congruences and normal filters

Let **A** be a residuated lattice and $a, x \in A$. We define the conjugates $\lambda_a(x) = [a\backslash(xa)] \wedge 1$ and $\rho_a(x) = ax/a \wedge 1$

An *iterated conjugate* is a composition $\gamma_{a_1}(\gamma_{a_2}(\dots\gamma_{a_n}(x)))$, where $n \in \omega$, $a_1, a_2, \dots, a_n \in A$ and $\gamma_{a_i} \in \{\lambda_{a_i}, \rho_{a_i}\}$, for all $i$

$F$ is a *filter* of **A** if it is a lattice filter, $a, b \in F$ implies $ab \in F$, and $1 \in F$

A filter is called *normal* if $x \in F$ implies $ax/a$, $a\backslash xa \in F$ for all $a \in A$ (i.e. it is closed under all conjugates; also called a deductive filter)

We will consider the correspondence between:

- Congruences on **A** and
- Normal filters of **A**

There are also correspondences with e.g. convex normal subalgebras



## Correspondence

If $R$ is a congruence on **A** and $F$ is a normal filter of **A** then

- $\mathbf{F}(R) = \uparrow[1]_R$ is a normal filter of **A** and
- $\mathbf{R}(F) = \{(a, b) \mid a\backslash b,\ b\backslash a \in F\}$ is a congruence of **A**

Normality of $\mathbf{F}(R)$: If $a \geq c \in [1]_R$ and $b \in A$ then

$$ba/b \geq (bc/b \wedge 1)\ R\ (b1/b \wedge 1) = 1$$

Compatibility of $\mathbf{R}(F)$: If $a\backslash b$, $b\backslash a \in F$ then

$(a \wedge c)(a\backslash b \wedge 1) \leq a(a\backslash b) \wedge c1 \leq b \wedge c$ so $(a \wedge c)\backslash(b \wedge c) \in F$ ($\vee$ same)

$a\backslash b \leq ca\backslash cb \in F$, $c\backslash(a\backslash b)c \leq ac\backslash bc \in F$ and $a\backslash b \leq (c\backslash a)\backslash(c\backslash b) \in F$

$a\backslash b \leq (a\backslash c)/(b\backslash c) \in F$ so $(b\backslash c)\backslash((a\backslash c)/(b\backslash c))(b\backslash c) \leq (a\backslash c)\backslash(b\backslash c) \in F$



## The normal filter lattice is isomorphic to Con(**A**)

The normal filters of **A** form a lattice, denoted by **NFil**(**A**)

This lattice is isomorphic to the lattice **Con**(**A**) via the maps **F** and **R**

Claim: **F** and **R** are inverse maps

$F = \uparrow[1]_{\mathbf{R}(F)}$: $a \in F$ implies $a \wedge 1 \in F$. Also $a \wedge 1 \leq 1$ implies $1 \leq (a \wedge 1)\backslash 1 \in F$, hence $a \wedge 1 \in [1]_{\mathbf{R}(F)}$.

$a \in \uparrow[1]_{\mathbf{R}(F)}$ implies $a \geq b \in [1]_{\mathbf{R}(F)}$, hence $1\backslash b = b \in F$ so $a \in F$.

$R = \mathbf{R}(\uparrow[1]_R)$: If $a\mathbf{R}(\uparrow[1]_R)b$, then $(a\backslash b) \wedge 1 \in [1]_R$ since $[1]_R$ is convex. Hence $a \vee b\ R\ a(a\backslash b \wedge 1) \vee b = b$, and similarly $a \vee b\ R\ a$, so $aRb$.

If $aRb$, then $1 = (a\backslash a \wedge b\backslash b \wedge 1)R(a\backslash b \wedge b\backslash a \wedge 1)$, hence $a\mathbf{R}(\uparrow[1]_R)b$

## Generation of normal filters

- The normal filter $F(X)$ of $\mathbf{A}$ generated by $X \subseteq A$ is $\uparrow\{\gamma_1(x_1)\gamma_2(x_2)\cdots\gamma_n(x_n) \mid n \geq 0, x_i \in X, \gamma_i \text{ iterated conjugates}\}$
- The congruence $R(Y)$ on $\mathbf{A}$ generated by $Y \subseteq A^2$ is $\mathbf{R}(F(\{a\backslash b \wedge b\backslash a \wedge 1 \mid (a,b) \in Y\}))$

For a finite residuated lattice any filter is generated by an idempotent $\leq 1$

An element $c$ is *central* if $cx = xc$ for all $x \in A$

The normal filters are the ones generated by a central idempotent $\leq 1$

$\mathbf{A}$ is subdirectly irreducible if it has a maximal central idempotent $< 1$

Exercise: A finite RL $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ with $\mathbf{B}$, $\mathbf{C}$ nontrivial if and only if there are central idempotents $b, c < 1$ such that $b \wedge c = \bot$ and $b \vee c = 1$



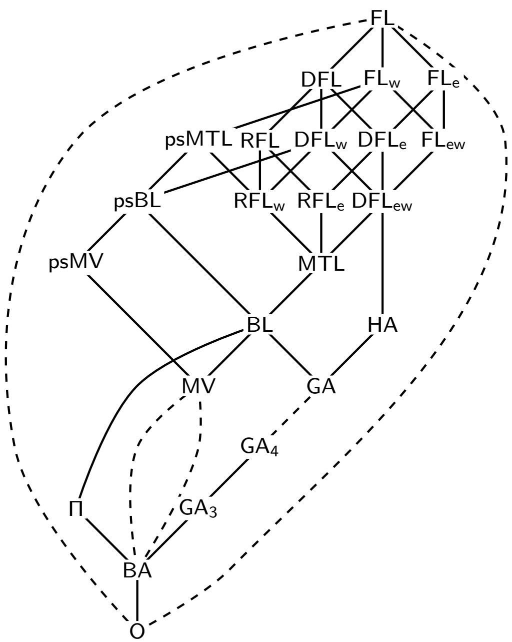
## Applications to subvarieties of residuated lattices

RL and FL denote the variety of residuated lattices and FL-algebras resp.

We view RL as the subvariety of FL axiomatized by $0 = 1$

Let's take a look at a map of the subvariety lattices $\Lambda_{\mathrm{FL}}$ and $\Lambda_{\mathrm{RL}}$

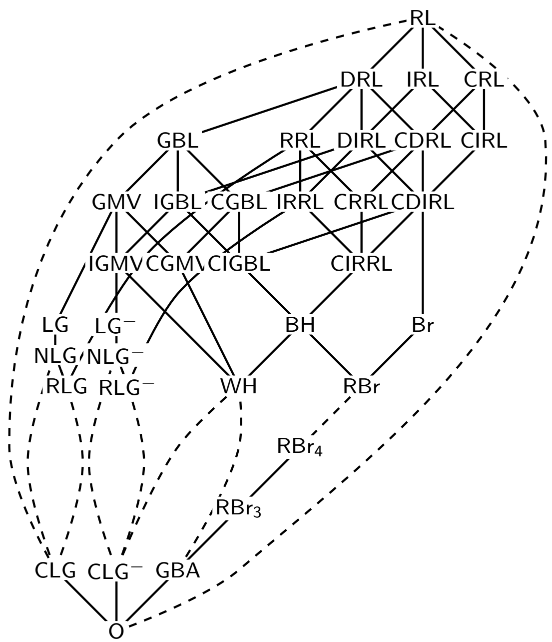





Some subvarieties of FL ordered by inclusion

D = distributive
w = bounded by 0, 1
e = commutative
R = representable
ps = pseudo
MTL = monoidal t-norm
BL = basic logic algebras
$x \wedge y = x(x \to y)$
HA = Heyting algebras
GA = Gödel algebras
MV = Multivalued algs
$x \vee y = (x \to y) \to y$
Π = Product algebras
BA = Boolean algebras
O = Trivial algebras






Some subvarieties of RL ordered by inclusion

D = distributive
I = integral $x \leq 1$
C = commutative
R = representable
GBL = generalized BL
$x \wedge y = x(x\backslash(x \wedge y))$ & mi
GMV = generalized MV
$x \vee y = x/((x \vee y)\backslash x)$& mi
BH = basic hoops
Br = Brouwerian algebras
WH = Wajsberg hoops
LG = lattice-ordered groups
N = normal valued
O = Trivial algebras

## Ultraproducts

$\mathcal{F}$ is a *filter over a set J* if $\mathcal{F}$ is a filter in $(\mathcal{P}(J), \subseteq)$

$\mathcal{F}$ defines a congruence on $\prod_{j\in J} \mathbf{A}_j$ via $a \equiv_{\mathcal{F}} b \Leftrightarrow \{j \in J | a(j) = b(j)\} \in \mathcal{F}$

$\prod_{j\in J} \mathbf{A}_j / \equiv_{\mathcal{F}}$ is called an *ultraproduct* if $J$ is a maximal filter

$\mathrm{P}_u\mathcal{K}$ is the class of all ultraproducts of members of $\mathcal{K}$

$\mathcal{K}$ is *finitely axiomatizable* if $\mathcal{K} = \mathrm{Mod}(E)$ for a finite set $E$

**Theorem**

*$\mathcal{K}$ and $\mathrm{P}_u\mathcal{K}$ satisfy the same first-order formulas*

*If $\mathcal{K}$ is a finite class of finite algebras then $\mathrm{P}_u\mathcal{K} = \mathcal{K}$*

Define $\mathrm{V}(\mathcal{K}) = \mathrm{HSP}(\mathcal{K})$, and recall this is a variety (i.e. closed via H, S, P)

Exercise: Show that $\mathrm{V}(\mathcal{K})$ is the smallest variety containing $\mathcal{K}$



## Congruence distributivity and Jónsson's Theorem

$\mathbf{A}$ is *congruence distributive* (CD) if $\mathrm{Con}(\mathbf{A})$ is a distributive lattice

A class $\mathcal{K}$ of algebras is *CD* if every algebra in $\mathcal{K}$ is CD

**Theorem (Jónsson 1967)**

*If $\mathcal{V} = \mathrm{V}(\mathcal{K})$ is congruence distributive then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathrm{HSP}_u\mathcal{K}$*

**Theorem**

*If $\mathcal{K}$ is a finite class of finite algebras and $\mathrm{V}(\mathcal{K})$ is CD then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathrm{HS}\mathcal{K}$*

*If $\mathbf{A}, \mathbf{B} \in \mathcal{V}_{\mathrm{SI}}$ are finite nonisomorphic and $\mathcal{V}$ is CD then $\mathrm{V}(\mathbf{A}) \neq \mathrm{V}(\mathbf{B})$*

$\mathcal{V}$ is *finitely generated* if $\mathcal{V} = \mathrm{V}(\mathcal{K})$ for some finite class of finite algebras

**Theorem**

*A finitely generated CD variety has only finitely many subvarieties*



## Lattices of subvarieties

$\mathbf{A}' = (A, f_{i_1}, f_{i_2}, \dots)$ is a *reduct* of $\mathbf{A} = (A, f_1, f_2, \dots)$ if $i_1 < i_2 < \cdots$ in $\mathbb{N}$

In this case $\mathbf{A}$ is called an *expansion* of $\mathbf{A}'$

Exercise: If $\mathbf{A}'$ is a reduct of $\mathbf{A}$ then $\mathrm{Con}(\mathbf{A})$ is a sublattice of $\mathrm{Con}(\mathbf{A}')$

The variety of lattices is CD, so any variety of lattice expansions is CD

Recall that for a variety $\mathcal{V}$ the lattice of subvarieties is denoted by $\Lambda_{\mathcal{V}}$

**Theorem**

*$\mathrm{HSP}_u(\mathcal{K} \cup \mathcal{L}) = \mathrm{HSP}_u\mathcal{K} \cup \mathrm{HSP}_u\mathcal{L}$ for any classes $\mathcal{K}, \mathcal{L}$*

*If $\mathcal{V}$ is CD then $\Lambda_{\mathcal{V}}$ is distributive and the map $\mathcal{V} \mapsto \mathcal{V}_{\mathrm{SI}}$ is a lattice embedding of $\Lambda_{\mathcal{V}}$ into $\mathcal{P}(\mathcal{V}_{\mathrm{SI}})$ (unless $\mathcal{V}_{\mathrm{SI}}$ is a proper class)*



## Size of the lattice of subvarieties

A set $S$ is *countable* if it there is a one-to-one function $f : S \to \mathbb{N}$

The subvariety lattices of HA (Heyting algebras) and Br (Brouwerian algebras) are uncountable, hence so are $\mathbf{\Lambda}_{\mathrm{FL}}$ and $\mathbf{\Lambda}_{\mathrm{RL}}$.

We will

- determine the size of the set of atoms in $\mathbf{\Lambda}_{\mathrm{FL}}$.
- outline a method for finding axiomatizations of certain varieties
- give a description of joins in $\mathbf{\Lambda}_{\mathrm{FL}}$.

## Boolean Algebras and **2**

The variety BA of Boolean algebras is generated by the 2-element alg. **2**

I.e. $\mathsf{BA} = \mathsf{HSP}(\mathbf{2}) = \mathsf{V}(\mathbf{2})$

Proof idea: In a distibutive lattice, any maximal proper congruence has two classes: a (prime) filter and a (prime) ideal, and each pair of distinct elements can be separated by such a congruence (the prime ideal-filter theorem for distributive lattices)

Then show that every BA has a subdirect decomposition with copies of **2**

So $\mathsf{BA} = \mathsf{SP}(\mathbf{2}) = \mathsf{HSP}(\mathbf{2})$

Now, $\mathsf{HSP_U}(\mathbf{2}) = \{\mathbf{2}, \mathbf{1}\}$, hence $(\mathsf{V}(\mathbf{2}))_{SI} = \{\mathbf{2}\}$



## Finitely generated atoms

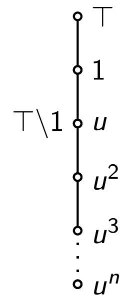
Consider the residuated chain $\mathbf{T}_n$ defined to the right

We also let $\top u = u\top = u$, hence $\top\backslash 1 = u$

Note that $\mathbf{T}_n$ is *strictly simple*, i.e. has no non-trivial subalgebras or homomorphic images

It follows that $\mathsf{V}(\mathbf{T}_n)$ is an atom of $\mathbf{\Lambda}_{\mathsf{RL}}$

Moreover, all these atoms are distinct so $\mathbf{\Lambda}_{\mathsf{RL}}$ has at least countably many atoms




$\mathbf{T}_n$

$\mathbb{Z}^- = \{n \in \mathbb{Z} \mid n \le 0\}$ is a cancellative RL with min, max, $+$ as operations



## Cancellative atoms

Left cancellativity ($ab = ac \;\Rightarrow\; b = c$) is equational: $x\backslash(xy) = y$

Right cancellativity is $(yx)/x = y$

CanRL denotes the variety of (left and right) cancellative RLs

**Theorem**

*There are only 2 cancellative atoms:* $\mathsf{V}(\mathbb{Z})$ *and* $\mathsf{V}(\mathbb{Z}^-)$

Let $\mathbf{L} \in \mathsf{CanRL}$. For $a \le 1$, we have $1 \le 1/a$.

*Claim*: If $a < 1$ and $1/a = 1$, then the subalgebra generated by $a$ is $\mathbb{Z}^-$

Since $a < 1$, we get $a^{n+1} < a^n$, for all $n \in \mathbb{N}$, by order preservation and cancellativity

Moreover, $a^{k+m}/a^m = a^k$ and $a^m/a^{m+k} = 1$, for all $m, k \in \mathbb{N}$



*Claim*: If for all $x < 1$, we have $1 < 1/x$, then $\mathbf{L}$ is an $\ell$-group.

For $a \in L$ set $x = (1/a)a$. Note that $x \le 1$, and if $x < 1$, then $1/x = 1/(1/a)a = (1/a)/(1/a) = 1$, cancellativity; so $x = 1$.

The construction of $\mathbb{Z}^-$ from $\mathbb{Z}$ actually works in general

The *negative cone* of a RL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ is the RL $\mathbf{A}^- = (A^-, \wedge, \vee, \cdot, \backslash^{\mathbf{A}^-}, /^{\mathbf{A}^-}, 1)$, where $A^- = \{a \in A : a \le 1\}$, $a\backslash^{\mathbf{A}^-} b = (a\backslash b) \wedge 1$ and $b/^{\mathbf{A}^-} a = (b/a) \wedge 1$.

## Idempotent atoms generated by chains

For $S \subseteq \mathbb{Z}$, we define

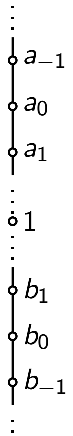
$a_i b_i = a_i$, if $i \in S$ and

$a_i b_i = b_i$, if $i \notin S$.

Although we may have

- $S \neq T$, but $\mathbf{N}_S \cong \mathbf{N}_T$
- $\mathbf{N}_S \not\cong \mathbf{N}_T$, but $\mathsf{V}(\mathbf{N}_S) = \mathsf{V}(\mathbf{N}_T)$
- $\mathsf{V}(\mathbf{N}_S)$ is not an atom

N. Galatos [2004] proved that there are continuum many atoms $\mathsf{V}(\mathbf{N}_S)$




$\mathbf{N}_S$



## Representable RL's

A residuated lattice is called *representable* if it is a subdirect product of linearly ordered RL's

RRL denotes the class of representable RL's

A linearly ordered RL satisfies the first-order formula $(\forall x, y)(x \leq y \text{ or } y \leq x)$, i.e., $[(\forall x, y)(1 \leq x \backslash y \text{ or } 1 \leq y \backslash x)]$

Representable Heyting algebras form a variety axiomatized by $1 = (x \to y) \vee (y \to x)$ (= Gödel algebras)

Representable commutative RL's form a variety axiomatized by $1 \leq (x \to y) \vee (y \to x)$

RRL is a variety axiomatized by $1 = \gamma_1(x \backslash y) \vee \gamma_2(y \backslash x)$

**Goal**: Given a class $\mathcal{K}$ of RL's axiomatized by a set of positive universal first-order formulas (PUF's), provide an axiomatization for $\mathsf{V}(\mathcal{K})$



## Joins of varieties

Recall that the meet of two varieties in $\mathbf{\Lambda}_{\mathsf{FL}}$ is their intersection

In fact, if $\mathcal{V}_1 = \mathrm{Mod}(E_1)$, $\mathcal{V}_2 = \mathrm{Mod}(E_2)$, then $\mathcal{V}_1 \wedge \mathcal{V}_2 = \mathrm{Mod}(E_1 \cup E_2)$

But the join of two varieties is the variety *generated* by their union

Also, if $\mathcal{V}_1$ is axiomatized by $E_1$ and $\mathcal{V}_2$ by $E_2$, then $\mathcal{V}_1 \vee \mathcal{V}_2$ is usually not axiomatized by $E_1 \cap E_2$.

Goals

- Find an axiomatization of $\mathcal{V}_1 \vee \mathcal{V}_2$ in terms of $E_1$ and $E_2$
- Find situations where: if $E_1$ and $E_2$ are finite, then $\mathcal{V}_1 \vee \mathcal{V}_2$ is finitely axiomatized



## Finite basis

[K. Baker 1977, B. Jónsson 1979] If $\mathcal{V}$ is a congruence distributive variety of finite type and $\mathcal{V}_{FSI}$ is strictly elementary, then $\mathcal{V}$ is finitely axiomatized.

Strictly elementary: Axiomatized by a single first-order sentence

Finitely SI: $id_{\mathbf{A}}$ is not the intersection of two non-trivial congruences

**Cor.** For every variety $\mathcal{V}$ of RL's, if $\mathcal{V}_{FSI}$ is *strictly* elementary, then the finitely axiomatized subvarieties of $\mathcal{V}$ form a lattice

**Pf.** For finitely axiomatized subvarieties $\mathcal{V}_1$, $\mathcal{V}_2$, $(\mathcal{V}_1 \vee \mathcal{V}_2)_{FSI} = (\mathcal{V}_1 \cup \mathcal{V}_2)_{FSI}$ is strictly elementary.

Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subvarieties of RL axiomatized by $E_1$, $E_2$, respectively, where $E_1$, $E_2$ have *no variables in common*.

The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the universal closure of (AND $E_1$) or (AND $E_2$), over infinitary logic, which is equivalent to the set $\{\forall\forall(\varepsilon_1 \text{ or } \varepsilon_2) : \varepsilon_1 \in E_1, \varepsilon_2 \in E_2\}$ of *positive universal first-order formulas* (PUFs).

## Finitely subdirectly irreducible RLs

In a RL, we say that 1 is *weakly join irreducible*, if for all negative $a, b$, whenever $1 = \gamma(a) \vee \gamma'(b)$, for all iterated conjugates $\gamma$, $\gamma'$, then $a = 1$ or $b = 1$

**Theorem (Galatos 2004)**

*A residuated lattice is FSI iff 1 is weakly join-irreducible*

Every PUF is equivalent to (the universal closure of) a disjunction of conjunctions of equations

$$s = t \iff (s \leq t \text{ and } t \leq s) \iff (1 \leq s\backslash t \text{ and } 1 \leq t\backslash s)$$



## Positive universal formulas (PUF's)

Every conjunction of equations $1 \leq p_i$ is equivalanent to the equation $1 \leq p_1 \wedge \cdots \wedge p_n$

So, every PUF is equivalent to a formula of the form

$$\alpha = \forall \bar{x} \ (1 \leq r_1 \text{ or } \cdots \text{ or } 1 \leq r_k)$$

Let $\widetilde{\alpha}_0$ be $(r_1 \wedge 1) \ \vee \ \cdots \ \vee \ (r_k \wedge 1) = 1$

Also, for $m > 0$ and $\aleph_0$ fresh variables $Y$, we define $\widetilde{\alpha}_m$ as the set of all equations of the form

$$\gamma_1 \ \vee \ \cdots \ \vee \ \gamma_k = 1$$

where $\gamma_i \in \Gamma^m_Y(r_i)$ for each $i \in \{1, \ldots, k\}$. Set $\widetilde{\alpha} = \bigcup_{n\in\omega} \widetilde{\alpha}_n$

Here $\Gamma^m_Y(a) = \{\pi_{y_1}\pi_{y_2} \cdots \pi_{y_m}(a \wedge 1) \mid y_i \in Y, \pi_{y_i} \in \{\lambda_{y_i}, \rho_{y_i}\}\}$



## PUF and equations

**Thm.** For a PUF $\alpha$ and a FSI RL $\mathbf{A}$, $\mathbf{A} \models \alpha$ iff $\mathbf{A} \models \widetilde{\alpha}$. **Pf.** ($\Rightarrow$) If $\bar{a}$ are elements in $A$, then $1 \leq r_i(\bar{a})$ for some $i$.
So, $\gamma(r_i(\bar{a})_{\wedge 1}) = 1$, for all $\gamma$; hence, $\gamma_1(r_1(\bar{a})_{\wedge 1}) \ \vee \ \cdots \ \vee \ \gamma_k(r_k(\bar{a})_{\wedge 1}) = 1$.

($\Leftarrow$) We have $1 = \gamma_1(r_1(\bar{a})_{\wedge 1}) \vee \ \cdots \vee \gamma_k(r_k(\bar{a})_{\wedge 1})$, for all $\gamma_i$.
Since $\mathbf{A}$ is FSI, 1 is weakly join irreducible, so $r_i(\bar{a})_{\wedge 1} = 1$, for some $i$; i.e., $r_i(\bar{a}) \leq 1$.

$\alpha = \forall \bar{x} \ (1 \leq r_1 \text{ or } \cdots \text{ or } 1 \leq r_k)$

$\widetilde{\alpha} = \{\gamma_1 \ \vee \ \cdots \ \vee \ \gamma_k \ = \ 1 \mid \gamma_i \in \Gamma_Y(r_i)\}$

**Theorem**

*Let $\mathcal{K}$ be a class of RLs axiomatized by a set $\Psi$ of PUF. Then $\mathsf{V}(\mathcal{K})$ is axiomatized, relative to* RL, *by $\widetilde{\Psi}$.*



**Proof.**

Let $\mathbf{A} \in \mathsf{RL}_{SI}$. By congruence distributivity and Jónsson's Lemma, $\mathbf{A} \in \mathsf{V}(\mathcal{K})$ iff $\mathbf{A} \in \mathsf{HSP_U}(\mathcal{K})$. Since PUFs are preserved under H, S and $\mathsf{P_U}$, $\mathbf{A} \in \mathsf{HSP_U}(\mathcal{K})$ iff $\mathbf{A} \in \mathcal{K}$. Finally, $\mathbf{A} \in \mathcal{K}$ iff $\mathbf{A} \models \Psi$ iff $\mathbf{A} \models \widetilde{\Psi}$ □

Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subvarieties of RL axiomatized by $E_1$, $E_2$, respectively, where $E_1$, $E_2$ have *no variables in common*.

The class $\mathcal{V}_1 \cup \mathcal{V}_2$ is axiomatized by the set of PUFs $\Psi = \{\forall\forall(1 \leq r_1 \text{ or } 1 \leq r_2) \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2\}$.

**Theorem**

*$\mathcal{V}_1 \vee \mathcal{V}_2$ is axiomatized by*

$$\widetilde{\Psi} = \{\gamma_1(r_1) \vee \gamma_2(r_2) = 1 \mid (1 \leq r_1) \in E_1, (1 \leq r_2) \in E_2, \gamma_i \in \Gamma\}$$

## Representable RLs

**Thm.** The variety RRL generated by all linearly ordered residuated lattices is axiomatized by the identity $\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1$

**Pf.** A RL is a chain iff it satisfies $\forall x, y(x \le y$ or $y \le x)$, or

$$\forall x, y(1 \le (x \vee y)\backslash x \text{ or } 1 \le (x \vee y)\backslash y).$$

Thus, RRL is axiomatized by the identities

$$1 = \gamma_1((x \vee y)\backslash x) \vee \gamma_2((x \vee y)\backslash y);\ \gamma_1, \gamma_2 \in \Gamma \qquad (\Gamma)$$

So, RRL satisfies the identity

$$\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1. \qquad (\lambda, \rho)$$

Conversely, the variety axiomatized by this identity satisfies

$$x \vee y = 1 \Rightarrow \lambda_z(x) \vee y = 1 \qquad x \vee y = 1 \Rightarrow x \vee \rho_w(y) = 1. \qquad \text{(imp)}$$

By repeated applications of (imp) on $(\lambda, \rho)$, we get $(\Gamma)$.



## Direct decompositions of $FL_w$-algebras

We now consider $FL_w$-algebras, i.e. residuated lattices with a 0 and 1 as the bottom and top element

Some of the results make no use of associativity, so they holds for integral bounded (residuated) groupoids ($ib(r)\ell$-groupoids for short)

An element $c$ in a $FL_w$-algebra $\mathbf{A}$ is *complemented* if there exists $c' \in A$ such that $c \wedge c' = 0$ and $c \vee c' = 1$.

The *Boolean center* of $\mathbf{A}$ is the set $B(\mathbf{A})$ of all complemented elements.

The direct decomposition results given here generalize similar results for MV-algebras [Cignoli, D'Ottaviano and Mundici 2000] and BL-algebras [Di Nola, Georgescu and Leustan 2000]

The first lemma is essentially due to [Birkhoff 1967].



### Lemma

*Let $\mathbf{A}$ be an $ib\ell$-groupoid and let $c \in B(\mathbf{A})$. Then $x \wedge c = xc = cx$ for all $x \in A$, and the Boolean center is a Boolean sublattice of central idempotent elements*

### Proof.

Suppose $A$ is an $ib\ell$-groupoid and $c \wedge d = 0$, $c \vee d = 1$. By integrality

$$xc \le x \wedge c = (x \wedge c)(c \vee d) = (x \wedge c)c \vee (x \wedge c)d \le xc \vee 0 = xc,$$

and similarly $cx \le c \wedge x \le cx$.

Suppose we also have $a \wedge b = 0$, $a \vee b = 1$. To see that $B(\mathbf{A})$ is a sublattice of $\mathbf{A}$, it suffices to show that $a \vee c$ and $b \wedge d$ are complements:

$(a \vee c) \wedge (b \wedge d) = (a \vee c)bd = abd \vee cbd = 0$ and

$(a \vee c) \vee (b \wedge d) = a \vee c \vee bd = a \vee c \vee bc \vee bd = a \vee c \vee b(c \vee d) = a \vee c \vee b = 1.$

Now $B(\mathbf{A})$ is complemented by definition, and it is a distributive lattice since $\cdot$ distributes over $\vee$, hence it is a Boolean lattice. □



## The boolean center is closed under residuals

### Lemma

*If $\mathbf{A}$ is a residuated $ib\ell$-groupoid then $B(\mathbf{A})$ is also closed under the residuals, the complement of $c$ is $-c = 0/c = c\backslash 0$ and $c\backslash x = x/c = -c \vee x$ for all $c \in B(\mathbf{A})$ and $x \in A$.*

### Proof.

For complements $c, d$ and any $x \in A$ we have

$$c\backslash x = (c \vee d)(c\backslash x) = c(c\backslash x) \vee d(c\backslash x) \le x \vee d$$

On the other hand $c(x \vee d) = cx \vee cd \le x$ implies $x \vee d \le c\backslash x$.

Hence $c\backslash x = d \vee x$, and for $x = 0$ we obtain $-c = c\backslash 0 = d$

Therefore $c\backslash x = -c \vee x$ for all $x \in A$

The results for $/$ follow similarly. □

For an $ib(r)\ell$-groupoid $\mathbf{A}$ and an element $c \in B(\mathbf{A})$, define

the *relativized subalgebra* $\mathbf{A}c = \downarrow c$ with unit $1^{\mathbf{A}c} = c$, operations $\wedge, \vee, \cdot$ restricted from $\mathbf{A}$,

and $a \backslash b = (a \backslash^{\mathbf{A}} b) \wedge c$, $a/b = (a/^{\mathbf{A}} b) \wedge c$ for all $a, b \in \downarrow c$.

**Lemma**

*For any $ib(r)\ell$-groupoid $\mathbf{A}$ and any $c \in B(\mathbf{A})$, the relativized subalgebra $\mathbf{A}c$ is an $ib(r)\ell$-groupoid*

**Proof.**

By the first lemma, $x \wedge c = xc = cx$, so $\mathbf{A}c$ has $c$ as a unit and is closed under $\wedge, \vee, \cdot$, hence it is an $ib\ell$-groupoid.
If $\mathbf{A}$ has residuals then for all $a, b, x \in \mathbf{A}c$ we have

$$ax \le b \quad \text{iff} \quad x \le^{\mathbf{A}} a \backslash^{\mathbf{A}} b \text{ and } x \le^{\mathbf{A}} c,$$

whence $a \backslash b = (a \backslash^{\mathbf{A}} b) \wedge c$, and similarly $a/b = (a/^{\mathbf{A}} b) \wedge c$. □



**Lemma**

*If $\mathbf{A}$ is an $FL_w$-algebra and $c \in B(\mathbf{A})$ then the map $f : \mathbf{A} \to \mathbf{A}c$ given by*

$$f(a) = ac \quad \text{is a surjective homomorphism,}$$

*hence $\mathbf{A}c$ satisfies all identities that hold in $\mathbf{A}$.*

**Proof.**

$f(1) = 1c = 1^{\mathbf{A}c}$, $(a \wedge b)c = a \wedge b \wedge c = ac \wedge bc$ and

$(a \vee b)c = ac \vee bc$ hence $f$ preserves $\wedge, \vee$.

If $\cdot$ is associative then $(ab)c = abcc = (ac)(bc)$.

In any residuated lattice $x \backslash y \le xz \backslash yz$, hence $f(a \backslash^{\mathbf{A}} b) \le f(a) \backslash f(b)$.

For the opposite inequality, we have $ac(ac \backslash bc) \le bc \le b$

and therefore $c(ac \backslash bc) \le a \backslash b$. □



## Complements in $\mathrm{FL}_w$ give direct decompositions

**Theorem**

*If $\mathbf{A}$ is an $FL_w$-algebra and if $c, d \in B(\mathbf{A})$ are complements then*

$$\mathbf{A} \cong \mathbf{A}c \times \mathbf{A}d$$

**Proof.**

Consider the map $h : \mathbf{A} \to \mathbf{A}c \times \mathbf{A}d$ defined by $h(a) = (ac, ad)$.

The preceding lemma shows that $h$ is a homomorphism, and

$h$ has an inverse given by $(x, y) \mapsto x \vee y$

since $ac \vee ad = a(c \vee d) = a1 = a$ and

for $x \le c$, $y \le d$ we have

$$((x \vee y)c, (x \vee y)d) = (xc \vee yc, xd \vee yd) = (x, y)$$

□



## Direct decompositions imply complements

Conversely, any direct decomposition of an $ib(r)\ell$-groupoid is obtained in this way, since the elements $(0, 1)$, $(1, 0)$ are complements.

**Corollary**

*A $FL_w$-algebra is directly indecomposable iff its Boolean center contains only the elements $\{0, 1\}$.*

The structure of directly indecomposables can be further analyzed by using *subdirect decompositions*

However, we now consider a *poset product* that can even be used to decompose subdirectly irreducible algebras

## Poset products

The *poset product* uses a partial order on the index set to define a subset of the direct product.

Specifically, let $\mathbf{X} = (X, \leq)$ be a poset, and assume $\{\mathbf{A}_i \mid i \in X\}$ is a family of algebras that have two constant operations denoted $0, 1$.

The poset product of $\{A_i \mid i \in X\}$ is

$$\prod_{\mathbf{X}} A_i = \{f \in \prod_{i\in X} A_i \mid \quad f(i) = 0 \text{ or } f(j) = 1 \quad \text{for all } i < j \text{ in } X\}$$

If $\mathbf{X}$ is an *antichain* then the poset product is the same as the direct product

If $\mathbf{X}$ is a *chain* and the $A_i$ are ordered, then the poset product is the (amalgamated) ordinal sum of the $A_i$



For an $\ell$-groupoid $\mathbf{A}$ define $I_{\mathbf{A}} = \{c \in A \mid c \wedge x = cx = xc \text{ for all } x \in A\}$

Note that $\wedge$ distributes over $\vee$ in $I_{\mathbf{A}}$, but $I_{\mathbf{A}}$ need not be a subalgebra of $\mathbf{A}$

A *GBL-algebra* is a residuated lattice that satisfies
$x \leq y \Rightarrow x = (x/y)y = y(y\backslash x)$

[J., Montagna 2006] prove that for bounded GBL-algebras, $\mathbf{I}_{\mathbf{A}}$ is a subalgebra, hence a Heyting algebra contained in $\mathbf{A}$,

and $B(\mathbf{A})$ is the subalgebra of complemented elements of $\mathbf{I}_{\mathbf{A}}$.

For MV-algebras $\mathbf{I}_{\mathbf{A}} = B(\mathbf{A})$

**Lemma**

*Let $\mathbf{A}$ be a $FL_w$-algebra and let $a, b \in I_{\mathbf{A}}$ with $a \leq b$.*
*Then the interval $[a, b] = \{x \in A \mid a \leq x \leq b\}$ is a $FL_w$-algebra, with*
*$0 = a$, $1 = b$,*
*$\wedge, \vee, \cdot$ inherited from $\mathbf{A}$, and $x\backslash y = (x\backslash^{\mathbf{A}} y) \wedge b$, $x/y = (x/^{\mathbf{A}} y) \wedge b$.*
*If $\mathbf{A}$ is a GBL-algebra, then so is $[a, b]$.*



We now generalize the poset sum decomposition result of [J., Montagna 2006] from finite GBL-algebras to certain $FL_w$-algebras

**Theorem**

*Consider a $FL_w$-algebra $\mathbf{A}$ with a finite subalgebra $\mathbf{C}$ such that $C \subseteq I_{\mathbf{A}}$,*

*and let $\mathbf{X}$ be the dual of the partially ordered set of completely join irreducible elements of $\mathbf{C}$.*

*For $c \in X$, let $c_*$ denote the unique lower cover of $c$ in $\mathbf{C}$.*

*If $\mathbf{A}c = {\downarrow}c_* \oplus [c_*, c]$ for all $c \in X$ then $\mathbf{A} \cong \prod_{\mathbf{X}} [c_*, c]$.*

The condition $\mathbf{A}c = {\downarrow}c_* \oplus [c_*, c]$ is actually satisfied for many GBL-algebras



A GBL-algebra is *normal* if every filter is a normal filter

**Theorem (J., Montagna)**

*A Blok-Ferreirim decomposition for GBL-algebras: Every subdirectly irreducible normal integral GBL-algebra decomposes as the ordinal sum of an integral GBL-algebra and a linearly ordered integral GMV-algebra.*

A residuated lattice is *n-potent* if it satisfies $x^{n+1} = x^n$

[J., Montagna] prove that any $n$-potent GBL-algebra is commutative, hence normal, so e.g. any finite GBL-algebra is commutative

**Corollary**

*Suppose $\mathbf{A}$ is an integral normal GBL-algebra such that $I_{\mathbf{A}}$ is finite*

*Then $\mathbf{A}$ is isomorphic to a poset product of GMV-algebras*

## Conclusion

Residuated lattices have many motivating examples from algebra and logic

Residuation has powerful consequences for the generation of congruences

The structure theory of residuated lattices is still developing

Positive universal formulas can be translated to equations

For certain varieties (e.g. GBL) the structure theory is even better behaved

There are many further interesting results to be discovered



## Bibliography I

P. Bahls, J. Cole, N. Galatos, P. Jipsen and C. Tsinakis, *Cancellative Residuated Lattices*, Algebra Universalis, 50(1) (2003), 83-106.

F. Belardinelli, H. Ono and P. Jipsen, *Algebraic aspects of cut elimination*, Studia Logics 68 (2001), 1-32.

G. Birkhoff, "Lattice Theory", 3rd ed., AMS Colloquium Publications, Vol. 25, 1967.

K. Blount and C. Tsinakis, *The structure of residuated lattices*, Internat. J. Algebra Comput. 13 (2003), no. 4, 437–461.

S. Burris and H. P. Sankapannavar, "A Course in Universal Algebra", Springer-Verlag, 1981, online at www.math.uwaterloo.ca/∼snburris/.

R. Cignoli, I. D'Ottaviano and D. Mundici, "Algebraic foundations of many-valued reasoning", Kluwer Academic Publishers, Dordrecht, Boston, London 2000.

N. Galatos. *Equational bases for joins of residuated-lattice varieties*, Studia Logica 76(2) (2004), 227-240.

N. Galatos. *Minimal varieties of residuated lattices*, Algebra Universalis 52(2) (2005), 215-239.

N. Galatos and P. Jipsen. *Residuated frames and applications to decidability*, preprint.



## Bibliography II

N. Galatos, P. Jipsen, T. Kowalski and H. Ono. Residuated Lattices: an algebraic glimpse at substructural logics, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

N. Galatos, J. Olson and J. Raftery, *Irreducible residuated semilattices and finitely based varieties*, Reports on Mathematical Logic 43 (2008), 85-108.

N. Galatos and H. Ono. *Algebraization, parameterized local deduction theorem and interpolation for substructural logics over FL*, Studia Logica 83 (2006), 279-308.

N. Galatos and H. Ono. *Glivenko theorems for substructural logics over FL*, Journal of Symbolic Logic 71(4) (2006), 1353-1384.

N. Galatos and H. Ono. *Cut elimination and strong separation for non-associative substructural logics*, manuscript.

J. Hart, L. Rafter and C. Tsinakis, *The Structure of Commutative Residuated Lattices*, Internat. J. Algebra Comput. 12 (2002), no. 4, 509-524.

P. Jipsen and C. Tsinakis. *A survey of residuated lattices*, in "Ordered Algebraic Structures" (J. Martinez, editor), Kluwer Academic Publishers, Dordrecht, 2002, 19-56.

P. Jipsen, *An overview of generalized basic logic algebras*, Neural Network World, 13(5) (2003), 491-500



## Bibliography III

P. Jipsen, *From semirings to residuated Kleene lattices*, Studia Logica 76(2) (2004), 291-303

P. Jipsen and F. Montagna, *On the structure of generalized BL-algebras*, Algebra Universalis, 55 (2006), 226-237

P. Jipsen and F. Montagna, *The Blok-Ferreirim theorem for normal GBL-algebras and its application*, Algebra Universalis, 60, to appear

P. Jipsen, *Generalizations of Boolean products for lattice-ordered algebras*, Annals of Pure and Applied Logic, to appear

H. Ono, *Semantics for substructural logics, Substructural logics*, 259–291, Stud. Logic Comput., 2, Oxford Univ. Press, New York, 1993.

K. Terui, *Which Structural Rules Admit Cut Elimination? - An Algebraic Criterion*, JSL 72(3) (2007), 738–754.

M. Ward and R. P. Dilworth, *Residuated Lattices*, Transactions of the AMS 45 (1939), 335–354.