Tutorial on Universal Algebra

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Outline

- Aim: cover the main concepts of universal algebra
- This is a *tutorial*
- Slides give definitions and results, few proofs
- Learning requires doing, so participants get individual exercises
- They vary in difficulty
- Part I: basic universal algebra and lattice theory
- Part II: some advances of the last 3 decades

Survey

How many of the following books are freely available for downloading?

0 **1** **2** **3** **4** **5** **6** **7** = Not sure

Stan Burris and H. P. Sankapannavar, "**A Course in Universal Algebra**", Springer-Verlag, 1981

David Hobby and Ralph McKenzie, "**The Structure of Finite Algebras**", Contemporary Mathematics v. 76, American Mathematical Society, 1988

Ralph Freese and Ralph McKenzie, "**Commutator theory for congruence modular varieties**", Cambridge University Press, 1987

Jarda Jezek, "**Universal Algebra**", 2008

Peter Jipsen and Henry Rose, "**Varieties of Lattices**", Lecture Notes in Mathematics 1533, Springer-Verlag, 1992

Keith Kearnes and Emil Kiss, "**The shape of congruence lattices**", 2006

ALL of them

Stan Burris and H. P. Sankapannavar, "**A Course in Universal Algebra**", Springer-Verlag, 1981, online at `www.math.uwaterloo.ca/~snburris`

David Hobby and Ralph McKenzie, "**The Structure of Finite Algebras**", Contemporary Mathematics v. 76, American Mathematical Society, 1988, online at `www.ams.org/online_bks/conm76/`

Ralph Freese and Ralph McKenzie, "**Commutator theory for congruence modular varieties**", Cambridge University Press, 1987, online at `www.math.hawaii.edu/~ralph/Commutator/comm.pdf`

Jarda Jezek, "**Universal Algebra**", 2008, online at `www.karlin.mff.cuni.cz/~jezek/ua.pdf`

Peter Jipsen and Henry Rose, "**Varieties of Lattices**", Lecture Notes in Mathematics 1533, Springer-Verlag, 1992, `www.chapman.edu/~jipsen`

Keith Kearnes and Emil Kiss, "**The shape of congruence lattices**", 2006, online at `spot.colorado.edu/~kearnes/Papers/cong.pdf`

Algebras and subalgebras

An n-ary *operation* on a set A is a function $f : A^n \to A$

0-ary operations are *constants* (fixed elements of A)

An *algebra* $\mathbf{A} = (A, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \ldots)$ is a set A with operations $f_i^{\mathbf{A}}$ of arity n_i

Superscript $^{\mathbf{A}}$ is useful when there are several algebras, otherwise omitted

The *signature* of an algebra is the list of arities $(n_1, n_2, \ldots)$

E.g. a *group* $\mathbf{G} = (G, \cdot, ^{-1}, 1)$ is an algebra of signature $(2, 1, 0)$

$g = f|_B$ means for all $b_i \in B$, $g(b_1, \ldots, b_n) = f(b_1, \ldots, b_n)$

$\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and $f_i^{\mathbf{B}} = f_i^{\mathbf{A}}|_B$ (all i)

i.e. B is closed under all operations of $\mathbf{A}$

Homomorphisms and isomorphisms

Let $\mathbf{A}$, $\mathbf{B}$ be algebras of the same signature

A *homomorphism* $h : \mathbf{A} \to \mathbf{B}$ is a function $h : A \to B$ such that for all i

$$h(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i}) = f_i^{\mathbf{B}}(h(a_1), \ldots, h(a_{n_i}))$$

h is *onto* if $h[A] = \{h(a) \mid a \in A\} = B$

In this case $\mathbf{B} = h[\mathbf{A}]$ is called a *homomorphic image* of A

h is *one-to-one* if for all $x, y \in A$, $x \neq y$ implies $h(x) \neq h(y)$

h is an *isomorphism* if h is a one-to-one and onto homomorphism

In this case $\mathbf{A}$ is said to be *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$

Products and HSP

$f : J \to \bigcup_{j \in J} A_j$ is a *choice function* if $f(j) \in A_j$ for all $j \in J$

The *cartesian product* $\prod_{j \in J} A_j$ is the set of all choice functions

The *direct product* of algebras $\mathbf{A}_j$ $(j \in J)$ is $\mathbf{A} = \prod_{j \in J} \mathbf{A}_j$ where $A = \prod_{j \in J} A_j$ and $f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})(j) = f_i^{\mathbf{A}_j}(a_1(j), \ldots, a_{n_i}(j))$ for all $j \in J$

Let $\mathcal{K}$ be a class of algebras of the same signature

$H\mathcal{K}$ is the class of *homomorphic images* of members of $\mathcal{K}$

$S\mathcal{K}$ is the class of algebras isomorphic to *subalgebras* of members of $\mathcal{K}$

$P\mathcal{K}$ is the class of algebras isomorphic to *direct products* of members of $\mathcal{K}$

$\mathcal{K}$ is a *variety* if $H(\mathcal{K}) = S(\mathcal{K}) = P(\mathcal{K}) = \mathcal{K}$ $\quad(\overset{\text{Tarski}}{\iff} HSP(\mathcal{K}) = \mathcal{K})$

Term algebras and equational classes

For a fixed signature, the *terms with variables from a set X* is the smallest set $T(X)$ such that $X \subseteq T(X)$ and

if $t_1, \ldots, t_{n_i} \in T(X)$ then "$f_i(t_1, \ldots, t_{n_i})$" $\in T(X)$ for all i

The *term-algebra over X* is $\mathbf{T}(X) = (T(X), f_1^{\mathbf{T}}, f_2^{\mathbf{T}}, \ldots)$ with

$f_i^{\mathbf{T}}(t_1, \ldots, t_{n_i}) =$ "$f_i(t_1, \ldots, t_{n_i})$" for all i and $t_1, \ldots, t_{n_i} \in T(X)$

An *equation* is a pair of terms (s, t) written "$s{=}t$"; often omit " "

An *assignment* into an algebra $\mathbf{A}$ is a homomorphism $h : \mathbf{T}(X) \to \mathbf{A}$

An algebra $\mathbf{A}$ *satisfies* $s{=}t$ if $h(s) = h(t)$ for all assignments into $\mathbf{A}$

For a set E of equations, $\mathrm{Mod}(E) = \{\mathbf{A} \mid \mathbf{A}$ satisfies $s{=}t$ for all $s{=}t \in E\}$

An *equational class* is of the form $\mathrm{Mod}(E)$ for some set of equations E

Varieties and equational logic

HSP "preserves" equations, so every equational class is a variety

Theorem (Birkhoff 1935)

Every variety is an equational class

For a class $\mathcal{K}$ of algebras $\mathrm{Eq}(\mathcal{K}) = \{s{=}t \mid \mathbf{A}$ satisfies $s{=}t$ for all $\mathbf{A} \in \mathcal{K}\}$

An *equational theory* is of the form $\mathrm{Eq}(\mathcal{K})$ for some class of algebras $\mathcal{K}$

$t[x{\mapsto}r]$ is the term t with all occurrences of x replaced by the term r

Theorem (Birkhoff 1935)

E is an equational theory if and only if for all terms q, r, s, t

$t{=}t \in E; \quad s{=}t \in E \implies t{=}s \in E; \quad r{=}s,\ s{=}t \in E \implies r{=}t \in E$

and $\quad q{=}r,\ s{=}t \in E \implies s[x{\mapsto}q]{=}t[x{\mapsto}r] \in E$

I.e. the rule of algebra: "replacing all x by equals in equals gives equals"

Examples of equational theories and varieties

A *binar* is an algebra $(A, \cdot)$ with one binary operation $x \cdot y$, written xy

A *semigroup* is an *associative* binar, i.e. satisfies $(xy)z = x(yz)$

A *band* is an *idempotent* semigroup, i.e. satisfies $xx = x$

A *semilattice* is a *commutative* band, i.e. satisfies $xy = yx$

A *unital binar* is an algebra $(A, \cdot, 1)$ that satisfies $1x = x$ and $x1 = x$

A *monoid* is a unital binar that is associative, i.e. a unital semigroup

$(A, \cdot, ^{-1}, 1)$ is a *group* if $\cdot$ is associative, $1x = x$ and $x^{-1} \cdot x = 1$

By Birkhoff's rules groups also satisfy $x1 = x$, $xx^{-1} = 1$ and $(x^{-1})^{-1} = x$

E.g. $x = 1x = (x^{-1})^{-1}x^{-1}x = (x^{-1})^{-1}1 = (x^{-1})^{-1}11 = (x^{-1})^{-1}x^{-1}x1 = 1x1 = x1$

More examples of equational theories and varieties

$\mathbf{A} = (A, \{m^{\mathbf{A}} : m \in M\})$ is an **M**-*set* for a monoid $\mathbf{M}$, if each $m^{\mathbf{A}}$ is a unary operation, $1^{\mathbf{A}} = \mathrm{id}_A$ and $m^{\mathbf{A}}(n^{\mathbf{A}}(x)) = (mn)^{\mathbf{A}}(x)$

If $\mathbf{M}$ is a group, then $\mathbf{A}$ is called a **G**-*set* (and $\mathbf{M}$ is denoted by $\mathbf{G}$)

$(S, +, \cdot)$ is a *semiring* if $(S, +)$ and $(S, \cdot)$ are semigroups and $x(y + z) = xy + xz$, $(x + y)z = xz + yz$ hold

Often $+$ is assumed *commutative* $(x + y = y + x)$ with *zero* $(x + 0 = x)$ and $x0 = 0 = 0x$ also $\cdot$ may have a unit $1x = x = x1$

$(A, +, 0, \{s : s \in S\})$ is an **S**-*module* for a semiring $\mathbf{S}$ if $(A, +, 0)$ is a (commutative) monoid, each s is a unary operation, $s(t(x)) = st(x)$, $s(x + y) = s(x) + s(y)$ and (if needed) $s(0) = 0$, $1(x) = x$

A semiring with $-$ is a *ring* if $(A, +, -, 0)$ is a commutative group, and a *field* if $(A \setminus \{0\}, \cdot, ^{-1}, 1)$ is also a commutative group (for some $^{-1}$)

A *vector space* over a field $\mathbf{F}$ is defined as an $\mathbf{F}$-module

Posets and meet-semilattices

A *poset* $(A, \leq)$ is a set A with a *partial order* $\leq$ (reflexive anti-sym. trans.)

For $S \subseteq A$ the *meet* $\bigwedge S$ is defined by $x \leq \bigwedge S \iff x \leq s$ for all $s \in S$

Note that $\bigwedge S$ is unique (if it exists; $\bigwedge$ = greatest lower bound)

$\bigwedge\{x, y\}$ is denoted by $x \wedge y$

A *meet-semilattice* is a poset in which $a \wedge b$ exists for all a, b

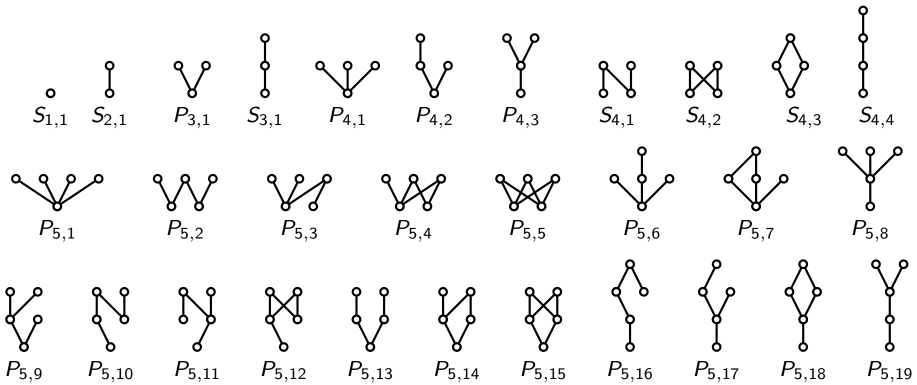
A meet-semilattice is *complete* if $\bigwedge S$ exists for all nonempty subsets S

Note that if $(A, \leq)$ is a meet-semilattice *then $(A, \wedge)$ is a semilattice* and for any semilattice, $ab = a \wedge b$ for the partial order $x \leq y \iff xy = x$

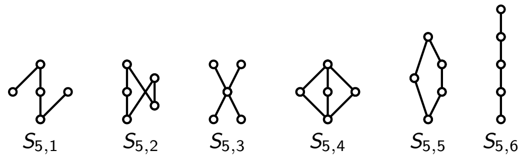
An element a has a *cover* b, denoted $a \prec b$, if $\{x \mid a < x \leq b\} = \{b\}$

A *Hasse diagram* of a poset has an upward line from dot a to b if $a \prec b$

(Dually)nonisomorphic connected posets with ≤ 5 elements



$P_{n,k}$ = nonselfdual
$S_{n,k}$ = selfdual
n-element poset



How many posets with 0 are **not** meet-semilattices?

Lattices

For a partial order $\leq$, define the *dual* $\geq$ by $\quad b \geq a \iff a \leq b$

$\mathbf{A}^\partial = (A, \geq)$ is the *dual poset* of $\mathbf{A} = (A, \leq)$

Every partial order concept has a dual, obtained by interchanging $\leq$ and $\geq$

The *join* $\bigvee$ is defined dually to the meet $\bigwedge$ (join = least upper bound)

A *join-semilattice* is a poset where $a \vee b = \bigvee\{a, b\}$ exists for all a, b

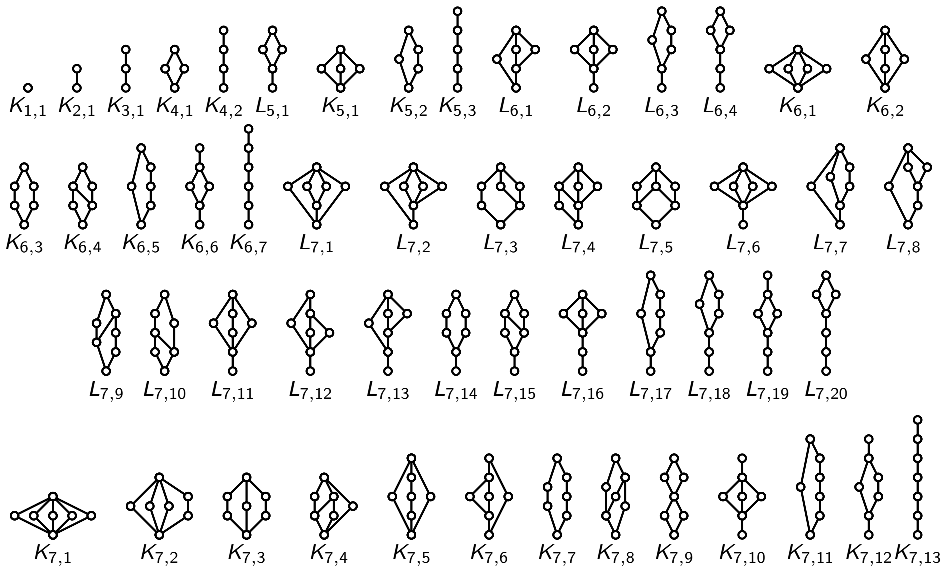
A *lattice* is a poset that is a meet-semilattice and a join-semilattice

Note that $x \leq y$ is definable by $x \vee y = y$, as well as by $x \wedge y = x$

Alternatively $\mathbf{A} = (A, \wedge, \vee)$ is a lattice iff $\wedge, \vee$ are associative, commutative and *absorbtive*, i.e. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

All (dually-)nonisomorphic lattices with ≤ 7 elements

$K_{n,k}$ = selfdual lattice of size n $\qquad L_{n,k}$ = nonselfdual lattices of size n



Examples of lattices

A lattice is *complete* if $\bigwedge S$ and $\bigvee S$ exist for all subsets S

A lattice is *bounded* if it has a top element $\top$ and a bottom element $\bot$

Note that *every complete lattice is bounded* and any *complete meet-semilattice with* $\top$ is a *complete lattice*

The *powerset* $\mathcal{P}(X)$ of all subsets of X is a complete lattice with $\bigcap, \bigcup$

The collection $\Lambda_\mathcal{V}$ of *subvarieties* of $\mathcal{V}$ is a complete lattice with $\bigwedge = \bigcap$

Any *linear order* (i.e. $x \leq y$ or $y \leq x$ for all x, y) is a lattice

A lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds ($\Leftrightarrow$ dual)

E.g. $\mathcal{P}(X)$ and any linear order are distributive lattices

Equivalence relations and congruences

Let $\mathbf{A}$ be an algebra and θ a binary relation on A

θ is an *equivalence relation* if it is reflexive, symmetric and transitive

θ is a *congruence* on $\mathbf{A}$ if it is an equivalence relation and

$x\theta y$ implies $f_i(a_1, \ldots, x, \ldots, a_n)\,\theta\, f_i(a_1, \ldots, y, \ldots, a_n)$ (all args, i)

The set $\mathrm{Con}(\mathbf{A})$ of *all congruences* on $\mathbf{A}$ is a complete lattice with $\bigwedge = \bigcap$

$\bot = id_A$ and $\top = A^2$; $\mathrm{con}(a, b) = \bigcap\{\theta \in \mathrm{Con}(\mathbf{A}) \mid a\theta b\}$

A *congruence class* or *block* is a set of the form $[a]_\theta = \{x \mid a\theta x\}$

$\{C_i : i \in I\}$ is a *partition* of A if $A = \bigcup_{i\in I} C_i$ and $C_i \cap C_j = \emptyset$ or $C_i = C_j$

The set A/θ of *all congruence classes* is a partition of A

Quiz

If $\mathbf{A} = (\{0, 1, 2\})$ is an algebra with no operations, how many congruences are there in $\mathrm{Con}(\mathbf{A})$?

1 **2** **3** **4** **5** **6** **0** = Not sure

If $\mathbf{C}_n = (\{0 < 1 < \cdots < n-1\}, \wedge, \vee)$ is a linearly ordered lattice, how many congruences are there?

1. $n - 1$ **2.** n **3.** 2^{n-1} **4.** 2^n **5.** 2^{2^n} **6.** Not sure

Homomorphic images and quotient algebras

The *quotient algebra* $\mathbf{A}/\theta = (A/\theta, f_1, f_2, \ldots)$ is defined by

$$f_i([a_1]_\theta, \ldots, [a_{n_i}]_\theta) = [f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})]_\theta$$

Note that f_i is *well-defined* if and only if θ is a *congruence*

For a homom. $h : A \to B$, define the *kernel* $\mathrm{ker}h = \{(x, y) \mid h(x) = h(y)\}$

Then $\mathrm{ker}h$ is a congruence on $\mathbf{A}$ and

the *natural map* $[\_]_\theta : \mathbf{A} \to \mathbf{A}/\theta$ is a homomorphism

Theorem (First Isomorphism Theorem)

$k : \mathbf{A}/\mathrm{ker}h \to h[\mathbf{A}]$ defined by $k([a]_{\mathrm{ker}h}) = h(a)$ is an isomorphism

Theorem (Second Isomorphism Theorem)

If $\theta \subseteq \psi$ are congruences on $\mathbf{A}$ and $\varphi = \{([a]_\theta, [b]_\theta) \mid a\psi b\}$ then $T \in Con(\mathbf{A}/\theta)$ and $(\mathbf{A}/\theta)/\varphi \cong \mathbf{A}/\psi$

Subdirectly irreducible algebras

An algebra is *directly decomposable* if it is isomorphic to a direct product of nontrivial algebras (happens rarely)

Let $\theta_j \in \mathrm{Con}(\mathbf{A})$ and define $h : \mathbf{A} \to \prod_{j\in J} \mathbf{A}/\theta_j$ by $h(a)(j) = [a]_{\theta_j}$

Then h is one-to-one if and only if $\bigcap_{j\in J} \theta_j = id_A$

In this case h is called a *subdirect decomposition* of $\mathbf{A}$

An element c in a complete lattice is *completely meet irreducible* if $c = \bigwedge S$ implies $c \in S$ for all subsets S; equiv. if $\uparrow c \setminus \{c\}$ is principal

$\mathbf{A}$ is *subdirectly irreducible* if id_A is completely meet irreducible in $\mathrm{Con}(\mathbf{A})$

Theorem (Birkhoff 1944)

Every algebra $\mathbf{A}$ has a subdirect decomposition using only subdirectly irreducible homomorphic images of $\mathbf{A}$

Varieties are generated by their s.i. members

Let $\mathcal{K}_{SI}$ be the class of subdirectly irreducible members of $\mathcal{K}$

Birkhoff's Theorem says that every algebra is a subalgebra of a product of subdirectly irreducible algebras (s.i. algebras for short)

So the s.i. algebras are *building blocks* of varieties:

$$\mathcal{V} = SP(\mathcal{V}_{SI})$$

For example the 2-element semilattice is the only s.i. semilattice and

the 2-element lattice is the only s.i. *distributive* lattice, hence

$$\text{Slat} = SP(S_2) \quad \text{and} \quad \text{DLat} = SP(C_2)$$

For any class of algebras $\mathcal{K}$, the *variety generated by* $\mathcal{K}$ is $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$
It is the smallest variety containing $\mathcal{K}$

Algebraic lattices

Let L be a complete lattice and $u \in L$

u is *compact* if $u \leq \bigvee X \Rightarrow u \leq x_1 \vee \cdots \vee x_n$ for some $x_1, \ldots, x_n \in X$

A complete lattice is *algebraic* if all element are joins of compact elements

For any algebra $\mathbf{A}$, $\text{Sub}(\mathbf{A})$ and $\text{Con}(\mathbf{A})$ are algebraic lattices

In this case, the compact elements are the finitely generated ones

Theorem (Birkhoff and Frink 1948)

Every algebraic lattice is isomorphic to the subalgebra lattice of some algebra

Theorem (Grätzer and Schmidt 1963)

Every algebraic lattice is isomorphic to the congruence lattice of some algebra

Lattices of varieties

$\theta \in \text{Con}(\mathbf{A})$ is *fully invariant* if $x\theta y \Rightarrow f(x)\theta f(y)$ for any endomorphism $f : \mathbf{A} \to \mathbf{A}$

Can rephrase Birkhoff's characterization of equational theories:

E is an equational theory iff E is a fully invariant congruence on the term algebra $\mathbf{T}(\omega)$

Also the set of fully invariant congruences is an *algebraic lattice* that is a *sublattice* of $\text{Con}(\mathbf{A})$

Note that $\quad E \subseteq \text{Eq}(\mathcal{K}) \iff \mathcal{K} \subseteq \text{Mod}(E)$

So Eq and Mod form a *Galois connection*, and the lattice of all equational classes is dually isomorphic to the lattice $\Lambda_{\mathcal{V}}$ of all varieties

For any variety $\mathcal{V}$, $\Lambda_{\mathcal{V}}$ is a *dually algebraic lattice* with the dually compact elements = varieties that are finitely based over $\mathcal{V}$

Filters and ideals in posets and lattices

For an element a in a poset, the *principal filter of* a is $\uparrow a = \{x \mid a \leq x\}$

A subset S of a poset is an *upset* if for all $a \in S$ we have $\uparrow a \subseteq S$

S is *down-directed* if for all $a, b \in S$ there is a $c \in S$ with $c \leq a$ and $c \leq b$

A *filter* F is a down-directed upset

An *ideal* is the dual concept of a filter, i.e. an *up-directed downset*

In a lattice, *down-directed* ⇔ *meet-closed* and *up-directed* ⇔ *join-closed*

An ideal or filter is *proper* if it is not the whole poset

An *ultrafilter* is a maximal (with respect to inclusion) proper filter

Ultraproducts

$\mathcal{F}$ is a *filter over a set I* if $\mathcal{F}$ is a filter in $(\mathcal{P}(I), \subseteq)$

$\mathcal{F}$ defines a *congruence* on $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ via $x \theta_{\mathcal{F}} y \iff \{i \in I : x_i = y_i\} \in \mathcal{F}$

$\mathbf{A}/\theta_{\mathcal{F}}$ is called a *reduced product*, denoted by $\prod_{\mathcal{F}} \mathbf{A}_i$

If $\mathcal{F}$ is an ultrafilter then $\mathbf{A}/\theta_{\mathcal{F}}$ is called an *ultraproduct*

$\mathsf{P}_u\mathcal{K}$ is the class of all ultraproducts of members of $\mathcal{K}$

Theorem

If $\mathcal{K} \models \phi$ then $\mathsf{P}_u\mathcal{K} \models \phi$ for any first order formula ϕ

If $\mathcal{K}$ is finitely based then the complement of $\mathcal{K}$ is closed under ultraproducts

If $\mathcal{K}$ is a finite class of finite algebras then $\mathsf{P}_u\mathcal{K} = \mathcal{K}$

Congruence distributivity and Jónsson's Theorem

$\mathbf{A}$ is *congruence distributive* (CD) if $\mathrm{Con}(\mathbf{A})$ is a distributive lattice

A class $\mathcal{K}$ of algebras is *CD* if every algebra in $\mathcal{K}$ is CD

Theorem (Jónsson 1967)

If $\mathcal{V} = \mathsf{V}\mathcal{K}$ is congruence distributive then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathsf{HSP}_u\mathcal{K}$

Corollary

If $\mathcal{K}$ is a finite class of finite algebras and $\mathsf{V}\mathcal{K}$ is CD then $\mathcal{V}_{\mathrm{SI}} \subseteq \mathsf{HS}\mathcal{K}$

If $\mathbf{A}, \mathbf{B} \in \mathcal{V}_{\mathrm{SI}}$ are finite nonisomorphic and $\mathcal{V}$ is CD then $\mathsf{V}\mathbf{A} \neq \mathsf{V}\mathbf{B}$

$\mathcal{V}$ is *finitely generated* if $\mathcal{V} = \mathsf{V}\mathcal{K}$ for some finite class of finite algebras

Corollary

A finitely generated CD variety has only finitely many subvarieties

Lattices of CD varieties

If $\mathcal{F}' \subset \mathcal{F}$ then $\mathbf{A}' = (A, (f^{\mathbf{A}} : f \in \mathcal{F}'))$ is a *reduct* of $\mathbf{A} = (A, (f^{\mathbf{A}} : f \in \mathcal{F}))$, and $\mathbf{A}$ is an *expansion* of $\mathbf{A}'$

Lemma

If $\mathbf{A}'$ is a reduct of $\mathbf{A}$ then $\mathrm{Con}(\mathbf{A})$ is a sublattice of $\mathrm{Con}(\mathbf{A}')$

The variety of lattices is CD, so any variety of algebras with lattice reducts is CD

For a variety $\mathcal{V}$ the lattice of subvarieties is denoted by $\Lambda_{\mathcal{V}}$

The meet is $\bigcap$ and the join is $\bigvee_{i \in I} \mathcal{V}_i = \mathsf{HSP}(\bigcup_{i \in I} \mathcal{V}_i)$

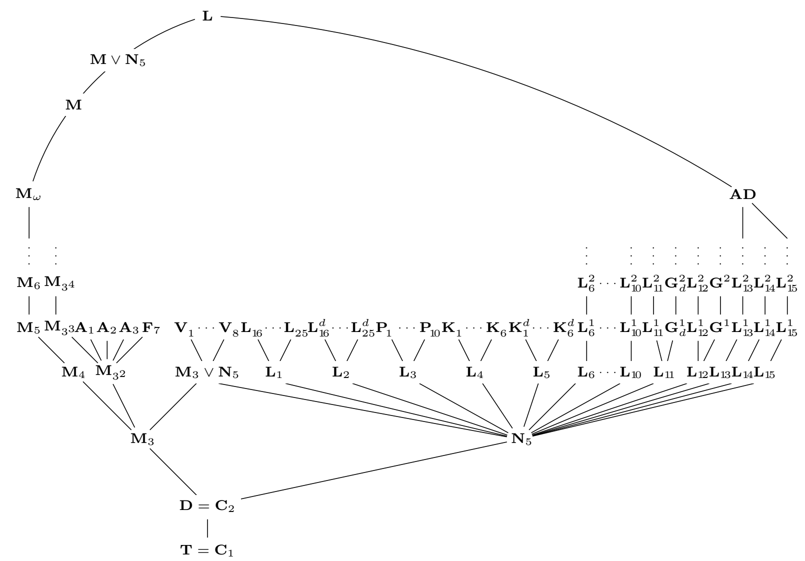
Corollary (Jónsson)

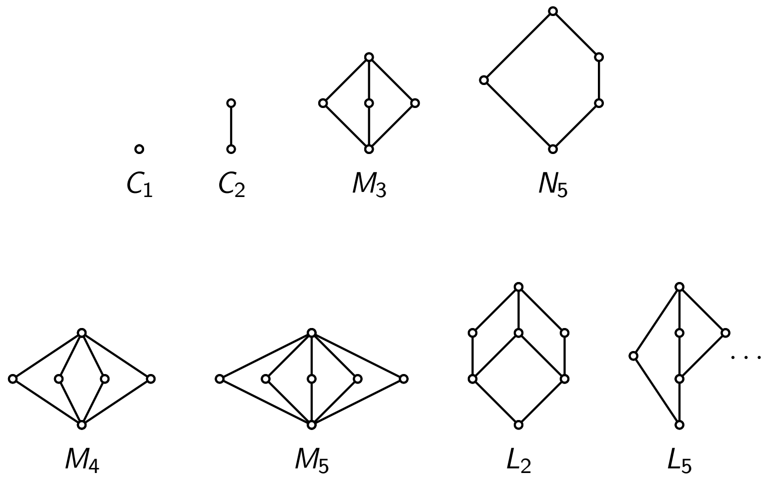
$\mathsf{HSP}_u(\mathcal{K} \cup \mathcal{L}) = \mathsf{HSP}_u\mathcal{K} \cup \mathsf{HSP}_u\mathcal{L}$ *for any classes $\mathcal{K}, \mathcal{L}$*

If $\mathcal{V}$ is CD then $\Lambda_{\mathcal{V}}$ is distributive and the map $\mathcal{V} \mapsto \mathcal{V}_{\mathrm{SI}}$ is a lattice embedding of $\Lambda_{\mathcal{V}}$ into "$\mathcal{P}(\mathcal{V}_{\mathrm{SI}})$" (unless $\mathcal{V}_{\mathrm{SI}}$ is a proper class)

Lattices of varieties provide a classification of equational theories

E.g. varieties of lattices:





Clearly it is very useful to know whether a variety is CD

If Con($\mathbf{A}$) is distributive, does it follow that HSP($\mathbf{A}$) is congruence distributive? 1. Yes 2. No 3. Not sure

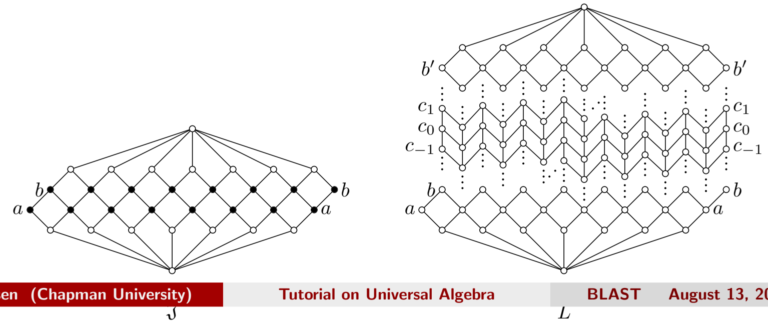
The finite height problem

By Jónsson's Lemma, any finitely generated CD variety has only finite many subvarieties

The converse can fail, making it harder to explore the bottom part of $\Lambda_{\mathcal{V}}$

The finite height problem for lattices asked whether every lattice variety with finitely many subvarieties is finitely generated

After 30 years, J. B. Nation (1996) found a counterexample



Free algebras

Let $\mathcal{K}$ be a class and let $\mathbf{F}$ be an algebra that is *generated* by a set $X \subseteq F$ (i.e. $\mathbf{F}$ has no proper subalgebra that contains X)

$\mathbf{F}$ is *$\mathcal{K}$-freely generated* by X if any $f : X \to \mathbf{A} \in \mathcal{K}$ extends to a homomorphism $\hat{f} : \mathbf{F} \to \mathbf{A}$ (the universal mapping property)

If also $\mathbf{F} \in \mathcal{K}$ then $\mathbf{F}$ is the *$\mathcal{K}$-free algebra on X* and is denoted by $\mathbf{F}_{\mathcal{K}}(X)$.

Lemma

If $\mathcal{K}$ is the class of all algebras (of a fixed signature) then the term algebra $\mathbf{T}(X)$ is the $\mathcal{K}$-free algebra on X

If $\mathcal{K}$ is any class of $\mathcal{F}$-algebras, let $\theta_{\mathcal{K}} = \bigcap\{\ker h \mid h : \mathbf{T}(X) \to \mathbf{A}$ is a homomorphism, $\mathbf{A} \in \mathcal{K}\}$. Then $\mathbf{F} = \mathbf{T}(X)/\theta_{\mathcal{K}}$ is $\mathcal{K}$-freely generated and if $\mathcal{K}$ is closed under subdirect products, then $\mathbf{F} \in \mathcal{K}$

$\Rightarrow$ free algebras exist in all (quasi)varieties (since they are S, P closed)

Examples of free algebras

A $\mathcal{K}$-free algebra on m generators satisfies precisely those equations with $\leq m$ variables that hold in all members of $\mathcal{K}$

$\mathbf{F}_{\mathsf{Sgrp}}(X) \cong \bigcup_{n\geq 1} X^n$ $\quad$ $\mathbf{F}_{\mathsf{Mon}}(X) \cong \bigcup_{n\geq 0} X^n$ $\quad$ $x \mapsto (x)$

These sets of n-tuples are usually denoted by X^+ and X^*

$\mathbf{F}_{\mathsf{Slat}}(X) \cong \mathcal{P}_{\text{fin}}(X) \setminus \{\emptyset\}$ $\quad$ $\mathbf{F}_{\mathsf{Slat}_1}(X) \cong \mathcal{P}_{\text{fin}}(X)$ $\quad$ $x \mapsto \{x\}$

$\mathbf{F}_{\mathsf{Srng}}(X) \cong \{$finite multisets of $X^*\}$

Lemma

If equality between elements of all finitely generated free algebras is decidable, then the equational theory is decidable

$\Rightarrow$ the equational theories of Sgrp, Mon, Slat, Slat$_1$, Srng are decidable

Free distributive lattices and Boolean algebras

$\mathrm{Sg}^{\mathbf{A}}(S)$ denotes the *subalgebra generated from the set S* in the algebra **A**

The free algebras for DLat and BA are also easy to describe

For a set X, let $h(x) = \{Y \in \mathcal{P}(X) : x \in Y\}$ and $x \mapsto h(x)$. Then

$\mathbf{F}_{\mathrm{DLat}}(X) \cong \mathrm{Sg}_{\mathrm{DLat}}^{\mathcal{P}(\mathcal{P}(X))}(h[X])$

$\mathbf{F}_{\mathrm{BA}}(X) \cong \mathrm{Sg}_{\mathrm{BA}}^{\mathcal{P}(\mathcal{P}(X))}(h[X])$

For finite X, the free BA is actually isomorphic to $\mathcal{P}(\mathcal{P}(X))$

For lattices, the free algebra on ≥ 3 generators is infinite but the equational theory is still decidable [Skolem 1928] (in polynomial time)

Birkhoff's construction of free algebras

Suppose $\mathcal{V} = \mathrm{V}(\mathcal{K})$ for a *set* $\mathcal{K}$ of algebras

Can assume $\mathcal{V} = \mathrm{V}\{\mathbf{A}\}$ where **A** is the product of algebras in $\mathcal{K}$

For a set X, consider the subalgebra **F** of $\mathbf{A}^{A^X}$ generated by $\{\pi_x : x \in X\}$ where $\pi_x(v) = v(x)$ (i.e. π_x projects onto the x-coordinate of v)

Then **F** has the universal mapping property for **A**: if $v : X \to A$ then $\hat{v} : \mathbf{F} \to \mathbf{A}$, defined by $\hat{v}(f) = f(v)$, is the projection homomorphism and it extends v since $\hat{v}(\pi_x) = \pi_x(v) = v(x)$

Since the universal mapping property is preserved by HSP, **F** is the free algebra

If **A** and X are finite, so is **F**

Hence every finitely generated free algebra in a f.g. variety is *finite*

But the size of $\mathbf{F}(n)$ can be doubly exponential (e.g. for Boolean algebras)

Often it is smaller, and to calculate **F** we may not have to use all of A^X

If $u, v : X \to A$ and $h : \mathbf{A} \to \mathbf{A}$ satisfies $h(u(x)) = v(x)$ for all $x \in X$ can delete v

This is called *thinning*

E.g. take **A** to be *your* 4-element monoid with f on the Exercise Sheet

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1
f	1	0	2	3

Try to calculate $\mathbf{F}_{\mathrm{V}(\mathbf{A})}(1)$ and (perhaps) $\mathbf{F}_{\mathrm{V}(\mathbf{A})}(2)$

Software useful for Universal Algebra

- GAP (Groups, Algebras, Programming) (gap-system.org)
- Prover9/Mace4 (prover9.org)
- Sage (sagemath.org) includes GAP and lots more
- Universal Algebra Calculator (uacalc.org)
- Lists of other packages http://orms.mfo.de/
- en.wikipedia.org/wiki/Comparison_of_computer_algebra_systems

Clones

A *clone* **C** is any set of operations on a set A that includes all projections $\pi_{ni}(x_1, \ldots, x_n) = x_i$ and is closed under composition:

$$f_0, \ldots, f_n \in \mathbf{C} \text{ implies } f_0(f_1(\ldots), \ldots, f_n(\ldots)) \in \mathbf{C}$$

Convenient to assume that nullary operations only appear as unary constant operations

$\mathrm{Clo}(A)$ is the clone of *all* operations on A

For an algebra **A**, a *term operation* is of the form $t^{\mathbf{A}}$ where t is a term

$\mathrm{Clo}(\mathbf{A})$ is the *clone* of all term operations on **A**

$\mathrm{Clo}_n(\mathbf{A})$ is the subset of all n-ary term operations on **A**

A and **B** are *term equivalent* if $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(\mathbf{B}')$ for some $\mathbf{B}' \cong \mathbf{B}$

Polynomial clones

A *polynomial operation* is of the form $t^{\mathbf{A}}(x_1, \ldots, x_m, a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in A$ are fixed (i.e. term-operations in the signature expanded with all unary constant operations)

$\mathrm{Pol}(\mathbf{A})$ is the *polynomial clone* of all polynominal operations on **A**

$\mathrm{Pol}_n(\mathbf{A})$ is the subset of all n-ary polynominal operations on **A**

A and **B** are *polynomially equivalent* if $\mathrm{Pol}(\mathbf{A}) = \mathrm{Pol}(\mathbf{B}')$ for some $\mathbf{B}' \cong \mathbf{B}$

E. Post (1941) showed there are countably many clones on a 2-element set

Only 7 of them are polynomial clones

There are *uncountably* many clones on a 3-element set

Simple algebras and the discriminator

A is *simple* if $\mathrm{Con}(\mathbf{A}) = \{\mathrm{id}_A, A^2\}$ i.e. has as few congruences as possible

Any simple algebra is subdirectly irreducible

A is a *discriminator algebra* if for some ternary term t
$\mathbf{A} \models x \neq y \Rightarrow t(x, y, z) = x$ and $t(x, x, z) = z$

Any subdirectly irreducible discriminator algebra is simple

$\mathcal{V}$ is a *discriminator variety* if $\mathcal{V}$ is generated by a class of discriminator algebras (for a fixed term t)

Discriminator varieties

A is *primal* if every operation on A is a term operation

E.g. the 2-element Boolean algebra and any prime field is primal

Theorem (Werner 1970)

A finite algebra **A** *is primal iff* **A** *has no proper subalgebras, no nontrivial automorphism and the ternary discriminator is a term operation of* **A**

Hence primal algebras generate discriminator varieties

McKenzie (1975) gave an equational characterization of discriminator varieties and proved that if such a variety has two constants that are distinct in each s.i. member then every universal formula φ can be translated into an equation $\hat{\varphi}$ such that

$$\varphi \text{ holds in all s.i. members iff } \hat{\varphi} \text{ holds in the variety}$$

This establishes a tight connection between first-order and equational logic

Unary discriminator in algebras with Boolean reduct

A *unary discriminator term* is a term d in an algebra $\mathbf{A}$ with a Boolean reduct such that $d(0) = 0$ and $x \neq 0 \Rightarrow d(x) = \top$

Theorem

An algebra with a Boolean reduct is a discriminator algebra iff it has a unary discriminator term

[Hint: let $d(x) = t(0, x, \top)^-$ and $t(x, y, z) = x \cdot d(x^- \cdot y + x \cdot y^-) + z \cdot d(x^- \cdot y + x \cdot y^-)^-$]

For a quantifier free formula ϕ we define a term ϕ^t inductively by $(r = s)^t = (r^- + s) \cdot (r + s^-)$, $(\phi \text{ and } \psi)^t = \phi^t \cdot \psi^t$, $(\neg\phi)^t = d((\phi^t)^-)$

Theorem

In a discriminator algebra with Boolean reduct $\phi \Leftrightarrow (\phi^t = 1)$

Mini algebraic logic excursion: Relation algebras

An *(abstract) relation algebra* is of the form $(A, +, 0, \cdot, \top, ^-, ;, 1, ^\smile)$ where

- $(A, +, 0, \cdot, \top, ^-)$ is a Boolean algebra
- $(A, ;, 1)$ is a monoid
- $(x;y) \cdot z = 0 \;\Leftrightarrow\; (x^\smile;z) \cdot y = 0 \;\Leftrightarrow\; (z;y^\smile) \cdot x = 0$

The last line states the Schröder equivalences (or DeMorgan's Thm K)

Lemma

In a relation algebra $x^{\smile\smile} = x$ and $^\smile$ is self-conjugated, i.e. $x^\smile \cdot y = 0 \Leftrightarrow x \cdot y^\smile = 0$. Hence $(x + y)^\smile = x^\smile + y^\smile$, $x^{-\smile} = x^{\smile -}$, $(x \cdot y)^\smile = x^\smile \cdot y^\smile$, $^\smile$ is an involution and $x;(y + z) = x;y + x;z$.

Hint: In a Boolean algebra $u = v$ iff $\forall x(u \cdot x = 0 \Leftrightarrow v \cdot x = 0)$

Theorem

A Boolean algebra expanded with an involutive monoid is a relation algebra iff $x;(y + z) = x;y + x;z$, $(x + y)^\smile = x^\smile + y^\smile$ and $(x^\smile; (x; y)^-) \cdot y = 0$

Concrete relation algebras

$\mathrm{Rel}(U) = (\mathcal{P}(U^2), \cup, \cap, \emptyset, U^2, ^-, ;, I_U, ^\smile)$ the *square relation algebra* on U

A *concrete relation algebra* is of the form $(\mathcal{C}, \cup, \cap, \emptyset, \top, ^-, ;, I_U, ^\smile)$ where $\mathcal{C}$ is a set of binary relations on a set U that is closed under the operations $\cup$, $^-$, $;$, $^\smile$, and contains I_U

Every square relation algebra is concrete.

Every concrete relation algebra is a relation algebra, and the largest relation is an equivalence relation

Relation algebras have applications in logic, set theory, combinatorics, program semantics, specification, databases, ...

Relation algebras are a discriminator variety

Let $\mathbf{A}a = (\downarrow a, +, 0, \cdot, a, ^{-_a}, ;_a, 1{\cdot}a, ^{\smile_a})$ be the *relative subalgebra* of relation algebra $\mathbf{A}$ with $a \in A$ where $x^{-_a} = x^-{\cdot}a$, $x;_a y = (x; y){\cdot}a$, and $x^{\smile_a} = x^\smile{\cdot}a$

An element a in a relation algebra is an *ideal element* if $a = \top;a;\top$

Theorem

$\mathbf{A}a$ is a relation algebra iff $a = a^\smile = a;a$

For any ideal element a the map $h(x) = (x{\cdot}a, x{\cdot}a^-)$ is an isomorphism from $\mathbf{A}$ to $\mathbf{A}a \times \mathbf{A}a^-$

A relation algebra is simple iff it is subdirectly irreducible iff it is not directly decomposable iff $0, \top$ are the only ideal elements iff $\top;x;\top$ is a unary discriminator term

Representable relation algebras

The class RRA of *representable relation algebras* is $\mathsf{SP}\{\mathsf{Rel}(X) : X \text{ is a set}\}$

Theorem

An algebra is in RRA iff it is embeddable in a concrete relation algebra

The class $\mathcal{K} = \mathsf{S}\{\mathsf{Rel}(X) : X \text{ is a set}\}$ *is closed under* H, S *and* $\mathsf{P_u}$

[*Outline:* $\mathsf{P_u S} \subseteq \mathsf{SP_u}$ *so if* $\mathbf{A} = \prod_{\mathcal{U}} \mathsf{Rel}(X_i)$ *for some ultrafilter* $\mathcal{U}$ *over* I, *let* $Y = \prod_{\mathcal{U}} X_i$, *define* $h : \mathbf{A} \to \mathsf{Rel}(Y)$ *by* $[x]h([R])[y] \Leftrightarrow \{i \in I : x_i R_i y_i\} \in \mathcal{U}$ *and show h is a well defined embedding*]

$\Rightarrow (\mathsf{V}\mathcal{K})_{\mathsf{SI}} \subseteq \mathcal{K}$ *by Jónsson's Theorem*

$\Rightarrow \mathsf{V}\mathcal{K} = \mathsf{SP}\mathcal{K} = \mathsf{RRA}$ *by Birkoff's subdirect representation theorem*

$\Rightarrow$ (Tarski 1955) RRA is a variety

Theorem

(Lyndon 1950) *There exist nonrepresentable relation algebras (i.e.* $\notin$ RRA*)*

(Monk 1969) RRA *is not finitely axiomatizable*

(Jonsson 1991) RRA *cannot be axiomatized with finitely many variables*

Outline of nonfinite axiomatizability: There is a sequence of finite relation algebras A_n with n atoms and the property that A_n is representable iff there exists a projective plane of order n

By a result of (Bruck and Ryser 1949) projective planes do not exist for infinitely many orders

The ultraproduct of the corresponding sequence of nonrepresentable A_n is representable, so the complement of RRA is not closed under ultraproducts

$\Rightarrow$ RRA is not finitely axiomatizable

Mal'cev conditions

Properties of an algebra are intimately connected with features of the congruence lattices

Mal'cev conditions postulate the existence of terms in an algebra that satisfy certain equations and characterize some (congruence) property

A *Mal'cev term* for a variety $\mathcal{V}$ is a ternary term p such that $p(x,x,y) = y = p(y,x,x)$ holds in $\mathcal{V}$

For any algebra with a group reduct, $p(x,y,z) = xy^{-1}z$ is a Mal'cev term

An algebra $\mathbf{A}$ is *congruence permutable (CP)* if $\theta \circ \psi = \psi \circ \theta$ for all $\theta, \psi \in \mathrm{Con}(\mathbf{A})$ $\quad (\Rightarrow \theta \vee \psi = \theta \circ \psi)$

A variety is CP if every member of it is CP

Theorem (Mal'cev 1954)

A variety $\mathcal{V}$ *is CP if and only if there exists a Mal'cev term for* $\mathcal{V}$

Congruence distributive varieties

A *majority term* for a variety $\mathcal{V}$ is a ternary term m such that $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ holds in $\mathcal{V}$

E.g. for an algebra with a lattice reduct $m(x,y,z) = (x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$

Recall a variety is CD if all members have *distributive* congruence lattices

Theorem

If there exists a Mal'cev term for $\mathcal{V}$ *then* $\mathcal{V}$ *is CD*

(Pixley 1963) A variety $\mathcal{V}$ *is CP and CD if and only if there exists a Mal'cev term and a majority term for* $\mathcal{V}$

A variety has *Jónsson terms* if there exist ternary terms $t_0, \ldots, t_n$ such that

$$t_0(x,y,z) = x \qquad t_i(x,y,x) = x \qquad t_n(x,y,z) = z$$
$$t_i(x,x,z) = t_{i+1}(x,x,z) \qquad \text{for even } i$$
$$t_i(x,z,z) = t_{i+1}(x,z,z) \qquad \text{for odd } i$$

Congruence distributive and congruence modular varieties

Theorem (Jónsson 1967)
A variety is CD if and only if it has Jónsson terms

A variety has *Day terms* if there exist 4-ary terms $q_0, \ldots, q_n$ such that

$$q_0(x,y,z,w) = x \qquad q_i(x,y,y,x) = x \qquad q_n(x,y,z,w) = w$$
$$q_i(x,y,y,z) = q_{i+1}(x,y,y,z) \quad \text{for even } i$$
$$q_i(x,x,z,z) = q_{i+1}(x,x,z,z) \quad \text{for odd } i$$

A lattice is *modular* if $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ holds

A variety is *CM* if all members have *modular* congruence lattices

Theorem (A. Day 1969)
A variety is CM if and only if it has Day terms

Further results

H. P. Gumm (1981) found another Mal'cev condition for congruence modularity that uses *ternary* terms

Also, if an algebra has permutable congruences then its congruence lattice is modular

There are many other Mal'cev condition for properties like n-permutability and congruence regularity (i.e. every congruence is determined by one of its blocks)

Congruence meet-semidistributive varieties

The congruences lattices of *semilattices* satisfy the *meet-semidistributive law* (D. Papert 1964):

$(\mathrm{SD}_\wedge)$ $\qquad x \wedge y = x \wedge z \Rightarrow x \wedge (y \vee z) = x \wedge y$

A variety is *CMSD* if $\mathrm{SD}_\wedge$ holds in the congruence lattices of all members

Bracket expressions $\mathbf{b} = b_0 \ldots b_n$ are defined *inductively* by:
() is a bracket expression, and if $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are bracket expressions then $(\mathbf{b}_1 \cdots \mathbf{b}_k)$ is a bracket expression

i, j are *matched* in $b_0 \ldots b_n$ if $b_i =$ (and $b_j =$) bound a subexpression

Theorem (R. Willard 2000)
A variety is CMSD iff there exists a bracket expression $\mathbf{b}$ *and ternary terms* $t_0, \ldots, t_n$ *such that* $t_0(x,y,z) = x$, $t_n(x,y,z) = z$
$t_i(x,x,z) = t_{i+1}(x,x,z)$ *for even* i, $t_i(x,z,z) = t_{i+1}(x,z,z)$ *for odd* i
$t_i(x,y,x) = t_j(x,y,x)$ *for each matched* i, j *in* $\mathbf{b}$

Finite basis problems

Mal'cev conditions establish congruence properties *for the whole variety*

They play a role in many results about which varieties can be defined using finitely many equations (i.e. are *finitely based*)

R. Lyndon (1951) showed that any 2-element algebra with finitely many operations generates a *finitely based variety*

The same holds for any *finite group* (Oates and Powell 1964), any *finite lattice*, even with additional operations (McKenzie 1970), any *finite ring* (Kruse, Lvov 1977), ...

Surprisingly, in 1954 Lyndon found a 7-element binar that generates a *nonfinitely based variety*, and Murskii (1965) found a 3-element example

K. Baker (1977) proved that any *finitely generated congruence distributive variety is finitely based* (using Jónsson terms)

Jónsson (1979) showed that "finitely generated" can be weakened to "*the finitely s.i. members are axiomatized by a first-order sentence*"

Further results

$\mathcal{V}$ has a *finite residual bound* if $\mathbf{A} \in \mathcal{V}_{SI} \Rightarrow |A| \leq n$ (some fixed n)

R. McKenzie (1987) proved that *congruence modular varieties with a finite residual bound* are finitely based (using commutator theory and Day terms)

R. Willard (2001) then proved the same result for *congruence meet-semidistributive varieties* (using his Mal'cev terms), and this was further generalized by Baker, McNulty, Wang (2005), also to *certain quasivarieties* by Maroti and McKenzie (2004)

Tarski's finite basis problem asked if it is decidable whether a given finite algebra generates a finitely based variety

After many decades, McKenzie (1996) proved that it is *undecidable*

Using similar techniques, McKenzie also proved that it is *undecidable* whether a *finite algebra generates a variety with finite residual bound*

Minimal algebras

Varieties are useful for classifying algebras of the same type

However it is very fine grained, and at the same time does not identify term-equivalent algebras

We now consider a highly successful classification of finite algebras, due to Hobby and McKenzie, based on 5 types associated with the *local structure* of an algebra

$\mathbf{A}$ is *minimal* if it is finite, $|A| \geq 2$ and every unary polynomial of $\mathbf{A}$ is a constant operation or a permutation on A

$\mathbf{A}$ and $\mathbf{B}$ are *polynomially equivalent* if $\mathrm{Pol}(\mathbf{A}) = \mathrm{Pol}(\mathbf{B}')$ for some $\mathbf{B}' \cong \mathbf{B}$

The types of minimal algebras

A minimal algebra $\mathbf{A}$ is of *type*

1 or *unary type* if it is polynomially equivalent to a **G**-*set*

2 or *affine type* if it is polynomially equivalent to a *vector space*

3 or *Boolean type* if it is polynomially equiv. to the *2-elmt Boolean algebra*

4 or *lattice type* if it is polynomially equivalent to the *2-element lattice*

5 or *semilattice type* if it is polynomially equiv. to the *2-elmt semilattice*

Theorem (Palfy 1984)

Every minimal algebra is associated with one of the above types

These 5 types already appear in the lattice of 2-element polynomial clones

Tame congruence theory in a nutshell

Where *else* do minimal algebras show up?

In *every finite algebra*, we just have to know where to look

$U \subseteq A$ is a *neighborhood of* $\mathbf{A}$ if $U = e(A)$ for some $e \in \mathrm{Pol}_1(\mathbf{A})$ that is *idempotent*, i.e., $e(e(x)) = e(x)$

The algebra *induced by* $\mathbf{A}$ *on* U is
$\mathbf{A}|_U = (U, \{f(x_1, \ldots, x_n)|_U : f \in \mathrm{Pol}(\mathbf{A}) \text{ and } f(U^n) \subseteq U\})$

$\mathrm{N}_{\mathbf{A}}(\theta)$ for $\mathrm{id}_A \prec \theta \in \mathrm{Con}(\mathbf{A})$ is the *set of neighborhoods that intersect at least one θ-block in ≥ 2 elements*

U is a *θ-minimal set* if U is a minimal member of $\mathrm{N}_{\mathbf{A}}(\theta)$

N is a *θ-trace* if $|N| \geq 2$ and $N = (\text{a } \theta\text{-minimal set}) \cap (\text{a } \theta\text{-block})$

The type set of an algebra

The *θ-body* of a θ-minimal set U is the union of all θ-traces in U

The *θ-tail* are the elements not in the θ-body

Theorem (Hobby McKenzie 1988)

For a finite algebra $\mathbf{A}$ *and* $\mathrm{id}_A \prec \theta \in \mathrm{Con}(A)$, *the algebra induced on any θ-trace is a minimal algebra, and any two such algebras are isomorphic*

This attaches a *type* to the edge below each minimal nonzero congruence

For the other edges do the same by factoring out each minimal congruence

For $\delta \prec \theta$ in $\mathrm{Con}(\mathbf{A})$ let $\mathrm{typ}(\delta, \theta)$ be the type of the minimal algebra

$$(\mathbf{A}/\delta)|_N/(\theta/\delta)|_N \quad \text{for any } \theta/\delta\text{-trace } N$$

The type-set of an algebra

$\mathrm{typ}(\mathbf{A}) = \{\mathrm{typ}(\alpha, \beta) : \alpha \prec \beta \text{ in } \mathrm{Con}(\mathbf{A})\}$ is the *type-set* of $\mathbf{A}$

$\mathrm{typ}(\mathcal{K}) = \bigcup\{\mathrm{typ}(\mathbf{A}) : \mathbf{A} \in \mathcal{K} \text{ and } \mathbf{A} \text{ is finite}\}$

$\mathbf{A}$ omits type i if $i \notin \mathrm{typ}(\mathbf{A})$

$\mathcal{K}$ omits type i if $i \notin \mathrm{typ}(\mathcal{K})$

Theorem

$\mathrm{V}(\mathbf{A})$ *is CM iff it omits type 1 (unary) and 5 (semilattice) and has no tails*

$\mathrm{V}(\mathbf{A})$ *is CD iff it also omits type 2 (affine) and has no tails*

Theorem (McKenzie Woods 2001)

For a finite algebra and each $i = 2, 3, 4, 5$ *there is no algorithm to decide whether type i occurs in* $\mathrm{V}(\mathbf{A})$

Many important topics were not touched upon in this tutorial, e.g. quasivarieties, commutator theory, congruence varieties, Boolean products, decidability results, term rewriting, ... (please consult references)

Currently there is much active research using tame congruence theory to attack the *dichotomy conjecture* related to the constraint satisfaction problem

Several lists of *open problems from past conferences* can be found on Miklos Maroti's website:

- Semigroups, algorithms and universal algebra, Louisville, 1998
- Posets and universal algebra, Vanderbilt, 2000
- Universal algebra, status of TCT problems, Budapest, 2001
- Universal algebra, Novi Sad, 2003
- Algebras, lattices and varieties, Colorado, 2004
- Constraint satisfaction problem, Oxford, 2006

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